• **Time-delay Spread – modulation effects.**

  The concepts of time delay spread and coherence bandwidth have been dealt with in the last Section for CW signals only, which is without considering the effect of modulation. Now we consider modulation effects.

  ➤ **Time-Delay Measurements**
  
  ➤ Outcome of channel sounding experiment of a time varying multipath channel using a very narrow transmitted pulse.

  Due to multipath nature of channel, received signal is a string of delayed echoes of narrow transmitted pulse.

  Due to the time varying nature of the channel, repeated soundings at different times produce different time spreaded echoes.

  We shall call the width or the spread of the echoes, the time delay spread or simply the delay spread of the channel.

  Consider now the transmitted signal is a sequence of bi-polar pulses, instead of a very narrow pulse.

  If the delay spread is a significant fraction (>50 %) of the symbol duration or even exceeds the symbol duration.

  Interference can occur between a given pulse and a delayed echo of an earlier pulse that has travelled via a longer path.

  This is known as inter-symbol interference (ISI).

  In Fig. below, we show a sequence of 4 transmitted bi-polar pulses.

  There are 3 paths in the channel producing 3 echoes at the receiver.

  In (b), the delay spread is < than 50% of \( T \), therefore the time window where there is no ISI is still sufficiently broad rendering a high probability of correct detection.

  In (c), the delay spread is > than 50% of \( T \), therefore the time window where there is no ISI has diminished rendering a high probability of wrong detection.
Delay Spread Function $h(t, \tau)$

- Transmitted signal $x(t)$

$$x(t) = A(t) \cos(\omega t + \phi(t)) = \Re\{u(t) \cdot e^{j\omega t}\}.$$  

$$u(t) = A(t) e^{j\phi(t)} \Rightarrow \text{complex envelope or low-pass equivalent signal} \Rightarrow \text{containing the modulation.}$$

- Received waveform from paths with delay $\tau_i \Rightarrow$

$$y_i(t) = \Re\{r_i(t) u(t - \tau_i) \exp[j\phi(t) - j\omega \tau_i] \exp j\omega t\}$$

- In above Eq., we have delayed $x(t)$ by $\tau_i$ and multiplied it with $r_i(t) \exp(j\phi(t))$.

- Relate $r_i(t) \exp(j\phi(t))$ to transmission coefficient $T_i(t) = T_c(t) + jT_s(t)$ on p. 11 and p. 12 of Lecture note #1.

$$T_c(t) = E_0 \sum_{n=1}^{N} C_n \cos(\omega_n t + \phi_n)$$

$$T_s(t) = E_0 \sum_{n=1}^{N} C_n \sin(\omega_n t + \phi_n)$$

$$T(t) = T_c(t) + jT_s(t) = r e^{j\theta}, \quad r = \sqrt{T_c^2 + T_s^2}, \quad \theta = \tan^{-1} \frac{T_s}{T_c}$$

- By multiplying $x(t)$ by $r_i(t) \exp(j\phi(t))$ we have in fact taken into account the superposition of many waves coming from all directions, but with delays of the order of $\tau_i$ grouped together.

- From $\omega_n = \beta V \cos \alpha_n = \frac{2\pi}{\lambda} V \cos \alpha_n = 2\pi f_m \cos \alpha_n$

- We can also see that we have included the effect of Doppler spread.
The received signal is then

\[ y(t) = \text{Re}[z(t)\exp(j\omega_c t)] \]

where

\[ z(t) = \sum_{i=1}^{\infty} r_i(t)u(t - \tau_i)\exp[j\theta_i(t) - j\omega_c \tau_i] \]

For a large number of paths, we can consider the received signal as a continuum of multipath components.

\[ z(t) = \int_0^\infty r(t, \tau)u(t - \tau)\exp(j\theta(t, \tau) - j\omega_c \tau)d\tau \]

\[ r(t, \tau) = \left| \sum_{\tau_i < \tau < \tau_i + \Delta\tau} r_i(t)\exp j\theta_i(t) \right| \]

\[ \theta(t, \tau) = \arg \left( \sum_{\tau_i < \tau < \tau_i + \Delta\tau} r_i(t)\exp j\theta_i(t) \right) \]

All \( \tau_i \) s between \( \tau \) and \( \tau + \Delta\tau \) are grouped together.

\[ z(t) = \int_0^\infty h(t, \tau)u(t - \tau)d\tau \]

\[ h(t, \tau) = r(t, \tau)\exp(j\theta(t, \tau) - j\omega_c \tau) \tag{1} \]

→ Delay Spread function or Impulse Response of channel.

→ Since \( h(t, \tau) = 0 \) for \( \tau \leq 0 \),

\[ z(t) = \int_{-\infty}^\infty h(t, \tau)u(t - \tau)d\tau. \]

- For a linear time invariant system

\[ u(t) \rightarrow \boxed{h(t)} \rightarrow z(t) \]

\[ z(t) = u(t) \ast h(t) \]

\[ = \int_{-\infty}^\infty u(\zeta)h(t - \zeta)d\zeta = \int_{-\infty}^\infty h(\tau)u(t - \tau)d\tau \]

\[ z(t) = u(t) \ast h(t) = \int_{-\infty}^\infty h(\zeta)u(t - \zeta)d\zeta = \int_{-\infty}^\infty u(\zeta)h(t - \zeta)d\zeta \tag{2} \]

\( h(t) \): Channel response to an impulse at \( t = 0 \).

\[ \delta(t) \rightarrow h(t) \]

\[ \delta(t - \zeta) \rightarrow h(t - \zeta) \]

- For a time-variant channel, the response to an impulse applied at \( t = \zeta \) will not have the same shape as the response to an impulse applied at \( t = 0 \).

- Introduce \( k(t, \zeta) \): channel response for an impulse applied at \( t = \zeta \).

From (2), instead of \( h(t - \zeta) \) we have \( k(t, \zeta) \), therefore

\[ z(t) = \int_{-\infty}^\infty u(\zeta)k(t, \zeta)d\zeta \]

now let \( \zeta = t - \tau \), \( d\zeta = -d\tau \)
\[ z(t) = \int_{-\infty}^{\infty} u(t-\tau)k(t,t-\tau) d\tau \]

Recalling from (1) that

\[ z(t) = \int_{-\infty}^{\infty} u(t-\tau)h(t,\tau) d\tau \]

\[ h(t,\tau) = k(t,t-\tau) \]

\[ \therefore h(t,\tau) \] : Channel response at \( t \) to an impulse applied at time \( t - \tau \),

that is applied at \( \tau \) seconds in the past.

Time variant channel response \( k(t,\xi) \) to unit impulses applied at

(a) \( t=0 \), (b) \( t=1 \), (c) \( t=\xi_0 \)
Impulse response in time and delay dimensions, (a) time variant channel $k(t, \xi)$, (b) time invariant channel, $h_{\text{inv}}(t - \xi)$.

- **Time-Variant Transfer Function**

- Now consider $U(f) \rightarrow$ Fourier transform of $u(t)$.

\[ u(t) = \int_{-\infty}^{\infty} U(f) e^{j2\pi ft} df. \]

Can consider this to be the summation of functions of form $e^{j2\pi ft}$.

- Response of channel to $e^{j2\pi ft}$ is from (1),

\[ \int_{-\infty}^{\infty} e^{j2\pi f(t-\tau)} h(t, \tau) d\tau = e^{j2\pi ft} H(f, t), \]

where $H(f, t) = \int_{-\infty}^{\infty} e^{-j2\pi f\tau} h(t, \tau) d\tau$.

$\rightarrow$ Fourier Transform of $h(t, \tau)$ with respect to the $\tau$ variable.

$\Rightarrow$ call $H(f, t)$ the Time-Variant Transfer Function.

Now,

\[
z(t) = \text{channel response of } u(t) = \int_{-\infty}^{\infty} U(f) H(f, t) e^{j2\pi ft} df
\]

$\rightarrow$ Same frequency domain technique as in time invariant linear channels except that $H(f, t)$ is a function of $t$. 

\[
\begin{align*}
U(f) & \rightarrow H(f) \rightarrow Z(f) = U(f)H(f) \\
U(t) & \rightarrow H(f, t) \rightarrow Z(f) = U(t)H(f, t)
\end{align*}
\]
• **Wide Sense Stationary Uncorrelated Scattering (WSS US) channel model**

  Mobile channel: \( h(t, \tau) \) linear time varying and random.
  - \( h(t, \tau) \) : Response at time \( t \) to an impulse input at time \( t - \tau \)
  - \( \tau \) is a delay variable.
  - Rate of change of \( h(t, \ldots) \) indicates the time-varying nature of channel.
  - \( H(f, t) \) : the Time-Variant Transfer Function is the Fourier Transform of \( h(t, \tau) \) with respect to the \( \tau \) variable.

  • **Auto-correlation Function of \( h(t, \tau) \) and \( H(f, t) \).**
    - \( h(t, \tau) \) and \( H(f, t) \) are stochastic or random processes.
    - We require their correlation functions.
      - \( R_{hh}(t_1, t_2; \tau_1, \tau_2) = \frac{1}{2} E[h(t_1, \tau_1) \cdot h^*(t_2, \tau_2)] \)
      - \( R_{HH}(f_1, f_2; t_1, t_2) = \frac{1}{2} E[H(f_1, t_1) \cdot H^*(f_2, t_2)] \)
    - A process is wide sense stationary (WSS) with \( t \) if its first two statistical moments, mean and auto correlation, are independent of absolute time.
      - \( \Rightarrow R_{hh}(\Delta t; \tau_1, \tau_2) \)
      - \( \Rightarrow R_{HH}(f_1, f_2; \Delta t) \).

  • **WSS Stationary (in \( t \))**
    - Correlation properties of \( h(t, \ldots) \) do NOT depend on the times of observations \( t_1, t_2 \) but only on their difference, \( \Delta t = t_1 - t_2 \).

  • **Uncorrelated Scattering (\( \tau \))**
    - Signal variations on paths arriving at different delays, i.e. \( h(\ldots, \tau_1), h(\ldots, \tau_2) \) are uncorrelated.
      - Together, \( \frac{1}{2} E[h(t_1, \tau_1)h^*(t_2, \tau_2)] = R_{hh}(t_1, t_2; \tau_1, \tau_2) = R_{hh}(\Delta t, \tau_1)\delta(\tau_1 - \tau_2) \).

  • **Power Delay Profile \( g(\tau) \)**
    - \( R_{hh}(\Delta t, \tau_1)\delta(\tau_1 - \tau_2) \) WSSUS channels.
      - \( g(\tau) = R_{hh}(0, \tau) = \frac{1}{2} E|H(f, \tau)|^2 \) is “Power Delay Profile” -- function describing the relative strength of multipath components at delays \( \tau \).
    - It is appropriate to normalize \( h(t, \tau) \) to have a unit power. The total power of the faded signal can be incorporated in the expression of the signal \( u(t) \).
      - \( \therefore \int_{-\infty}^{\infty} g(\tau) d\tau = 1 \)
    - A measure of the width of \( g(\tau) \) is the rms delay spread \( \tau_{rms} \).
      - \( \sigma_T = \tau_{rms} = \sqrt{\int (\tau - \tau_{mean})^2 g(\tau) d\tau} \)
      - where \( \tau_{mean} = \int \tau g(\tau) d\tau \)
• **Typical Power Delay Profiles**

  • **Exponential**

  \[
  g(\tau) = \frac{1}{\tau_{\text{rms}}} \exp \left( -\frac{\tau + \tau_{\text{rms}}}{\tau_{\text{rms}}} \right)
  \]

  \[
  \int_{-\tau_{\text{rms}}}^{\infty} g(\tau) d\tau = \tau_{\text{mean}}
  \]

  \[
  = \int_{-\tau_{\text{rms}}}^{\infty} \frac{\tau}{\tau_{\text{rms}}} \exp \left( -\frac{\tau + \tau_{\text{rms}}}{\tau_{\text{rms}}} \right) d\tau = 0.
  \]

  • \[ \int_{-\tau_{\text{rms}}}^{\infty} \tau^2 g(\tau) d\tau = \tau_{\text{rms}}^2. \]

  *Fig. Typical power delay profiles: One-sided exponential profile*

• **Double Spike**

  \[
  g(\tau) = \frac{1}{2} \delta(\tau + \tau_{\text{rms}}) + \frac{1}{2} \delta(\tau - \tau_{\text{rms}})
  \]

  \[
  \tau_{\text{mean}} = 0
  \]

  \[
  \int \tau^2 g(\tau) d\tau
  \]

  \[
  = \int \tau^2 \cdot \frac{1}{2} \delta(\tau + \tau_{\text{rms}}) + \delta(\tau - \tau_{\text{rms}}) d\tau
  \]

  \[
  = \frac{1}{2} \tau_{\text{rms}}^2 + \tau_{\text{rms}}^2 = \tau_{\text{rms}}^2
  \]

  *Fig. Typical power delay profiles: Double spike profile*
• **Frequency Domain Correlation**

Recall the Time-Variant Transfer Function:

$$H (f, t) = \int_{-\infty}^{\infty} h(t, \tau) e^{-j2\pi ft} d\tau$$

F.T. of \(h(t; \tau)\) in \(\tau\) (delay variable) is also WSS in \(t\) (due to WSS of \(h(t; \tau)\)).

\[
R_{HH} (f_1, f_2; t_1, t_2) = \frac{1}{2} E \left[ H(f_1; t_1) H^*(f_2; t_2) \right]
\]

\[
= \frac{1}{2} E \left[ \int h(t_1, \tau_1) h^*(t_2, \tau_2) e^{j2\pi (\tau_2 f_2 - \tau_1 f_1)} d\tau_2 d\tau_1 \right]
\]

\[
= \int R_{hh}(t_1, t_2; \tau_1, \tau_2) e^{j2\pi (\tau_2 f_2 - \tau_1 f_1)} d\tau_2 d\tau_1
\]

Then for WSSUS channels \(h(t; \tau)\)

- \(R_{hh}(t_1, t_2; \tau_1, \tau_2) = R_{hh}(\Delta t; \tau_1) \delta(\tau_1 - \tau_2)\)

\[
\therefore R_{HH}(f_1, f_2; t_1, t_2) = \int R_{hh}(\Delta t; \tau_1) \delta(\tau_1 - \tau_2) e^{j2\pi (\tau_2 f_2 - \tau_1 f_1)} d\tau_2 d\tau_1
\]

\[
\Delta f = f_1 - f_2
\]

- \(R_{HH}(\Delta f; 0) = \int R_{hh}(0; \tau_1) e^{-j2\pi \Delta f \tau_1} d\tau_1\)

- \(R_{HH}(\Delta f; 0) = \text{Fourier Transform of power - delay profile} = F[g(\tau)]\).

Frequency domain correlation: measured by transmitting on 2 frequencies, \(\Delta f\) apart, simultaneously, and measuring the amplitude correlation between the 2 received signals.

\(\Rightarrow R_{HH}(\Delta f)\) measures ‘frequency selectivity’ of multipath fading.
Example: Impulse response measurements in a typical urban environment give the delay distribution as

\[ p(\tau) = \frac{1}{\tau_r} e^{-\frac{\tau}{\tau_r}}, \tau \geq 0. \]

What would be the coherence bandwidth of such a mobile radio channel?

Solution:

\[ R_{HH}(\Delta f; 0) = \int R_{hh}(0; \tau) \exp(-j2\pi\Delta f \tau) d\tau \]

\[ = \int_0^{\infty} \exp\left(-\frac{\tau}{\tau_r}(1 + j2\pi\Delta f \tau_r)\right) \frac{d\tau}{\tau_r} = \frac{1}{1 + j2\pi\Delta f \tau_r}. \]

\[ |R_{HH}(\Delta f; 0)|^2 = \frac{1}{1 + (2\pi\Delta f \tau_r)^2}. \]

We may also define coherence bandwidth as that \( \Delta f \) for which

\[ |R_{HH}(\Delta f; 0)|^2 = 0.5. \]

In this case,

\[ \frac{1}{1 + (2\pi\Delta f \tau_r)^2} = 0.5 \quad \text{or} \quad (2\pi\Delta f \tau_r)^2 = 1 \quad 2\pi\Delta f = 1/\tau_r \]

Coherence bandwidth \( B_c = \Delta f = 1/(2\pi\tau_r) \) ⇒ the same result as in Lecture #2, p. 20, Eq. (28).

Channel Classification

- In this section since we have included modulation in the analysis, we can relate \( B_c \) and \( T_c \) to \( B_s \) and \( f_m \).
- Frequency flat, multiplicative (time selective fading).
- \( B_S << B_c \).
- where, \( B_s = \text{signal bandwidth}, B_c = \text{coherence bandwidth}. \)

\[ H(f, t) \]

- All frequency components in \( U(f) \) undergo the same attenuation and linear phase shift through the channel.
- \( H(f, t) \cong H(0, t) \) since \( U(f) \) has its frequency content concentrated in the vicinity of \( f = 0 \).
- Recall from page 13.

\[ z(t) = \int U(f) \cdot H(f, t) e^{j2\pi ft} df \]

\[ \cong H(0, t) \int U(f) e^{j2\pi ft} df \]

\[ = H(0, t) \cdot u(t) \]

\[ a_s(t) = H(0, t) = \int h(t, \tau) d\tau \]

\[ z(t) = a_s(t) \cdot u(t) \]

- If rate of change of \( a_s(t) \) with \( t \) is smaller than rate of change of \( u(t) \) with \( t \), then shape of signal pulse is preserved. However it undergoes amplitude fading whenever \( |a_s(t)| \) dips.
• **Frequency selective, time flat channels.**
  Received signal duration (time during which signal is in flight) less than coherence time. \( T_S < T_c \) → Channel appears to the signal as time invariant. \( H(f,t) → H(f) \) → Time flat channels.

• \( Z(f) = H(f)U(f) \). However frequency selectivity means \( B_S > B_c \), which implies that the signal spectrum \( U(f) \) will be modified by the multiplication with \( H(f) \). Shape of received waveform distorted.

• Also for digital transmission, \( T_s \sim \frac{1}{B_s} < \frac{1}{B_c} \sim \sigma_T = \tau_{rms} \) → Inter-Symbol-Interference (ISI).

---

**Classification of Multi-path Fading**

• **2 Channel parameters**

  (1) Multi-path (rms) spread / coherence BW
  \[
  \downarrow \quad \downarrow \quad \sigma_T \quad B_c
  \]
  captures the multi-path channel conditions via delay-spread (in-time) or amplitude correlation (in frequency).

  (2) Doppler spread / Coherence Time
  \[
  \downarrow \quad \downarrow \quad f_m \quad T_c
  \]
  \[
  f_m \sim \frac{1}{T_c}
  \]
  captures the rate of multi-path channel variations via spread of carrier (in frequency) or correlation of channel impulse response (in time).

• **Two System Design Parameters**

  (1) Symbol Period: \( T_s \)

  (2) Transmission BW: \( B_s \)

  **Narrowband** (PSK / QAM) : **Wideband** (Spread - Spectrum)

  \[
  B_s \approx \frac{1}{T_s} \quad : \quad B_s \approx \frac{1}{T_{chip}} >> \frac{1}{T_s}
  \]

  \[
  \overline{T_s} \quad \overline{T_{chip}}
  \]
• **Classification Based on Multipath Spread**

**Flat (Freq. Non-select Fading)** : **Freq.-Select Fading**

\[ B_s << B_c \] : \[ B_s >> B_c \]

<table>
<thead>
<tr>
<th>[ B_s ]</th>
<th>[ B_c ]</th>
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<tbody>
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<td>&lt;&lt;</td>
<td>&gt;&gt;</td>
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Transmitted pulse shape is (relatively) undisturbed, but amplitude fades with time.

• **For Narrowband Modulation**

\[ T_s \approx \frac{1}{B_s} \Rightarrow T_s >> \sigma_T \]

\[ T_s, \sigma \] : channel variations

\[ T_s, \sigma_T \] : Inter-symbol Interference (ISI)

\[ T_s << \sigma_T \Rightarrow (\text{no ISI}) \]

\[ T_s, \sigma_T \] : channel is approximately invariant over several symbol duration.

\[ T_s, \sigma_T \] : Faster than baseband signal variations.

\[ T_s, \sigma_T \] : Channel variation slower than baseband signal variation.

\[ \Rightarrow \text{Flat Fading} \longleftrightarrow \text{No Inter-symbol Interference (ISI).} \]

\[ \Rightarrow \text{Frequency Selective Fading} \]

\[ \Rightarrow \text{ISI} \]

\[ T_s << \sigma_T \Rightarrow \text{ISI} \]

\[ \text{Faster than baseband signal variations.} \]

\[ \Rightarrow \text{Channel variation slower than baseband signal variation,} \]

\[ \text{Channel is approximately invariant over several symbol duration.} \]

\[ \Rightarrow \text{For Wideband Modulation} \]

\[ \Rightarrow \text{Suppose } \sigma_T << T_s \text{ (no ISI)} \]

However, it is possible that

\[ B_s \approx \frac{1}{T_c} >> B_c \approx \frac{1}{\sigma_T} \]

But \[ B_s >> B_c \Rightarrow \text{Frequency-Selectivity.} \]

• **Classification Based on Doppler Spread**

**Fast Fading** : **Slow Fading**

\[ T_c << T_s \] : \[ T_c >> T_s \]

\[ \Rightarrow \text{Channel variations} \rightarrow \text{Channel variation slower} \]

\[ \text{Faster than baseband signal} \rightarrow \text{than baseband signal variation,} \]

\[ \text{Channel is approximately invariant} \rightarrow \text{over several symbol duration.} \]

\[ \Rightarrow \text{For wide-band signals, possible to have no ISI and frequency selectivity simultaneously.} \]