Maximum Diversity Order in Cooperative MAC for General Modulation Schemes

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Abstract—We consider two sources that cooperatively send their messages to a destination over Rayleigh-fading channels. Assuming a decode-and-forward scheme, we obtain conditions to achieve the maximum diversity order for a general class of modulation schemes. This class includes the BPSK modulation where the mutual information is bounded, and Gaussian signaling where the mutual information is unbounded. With mutual information that is bounded, clearly the maximum achievable rate is limited; interestingly, our results quantify the exact conditions for the diversity order to be fundamentally limited.

Index Terms—Cooperation diversity, cooperative multiple-access channel, practical modulations, BPSK modulation.

I. INTRODUCTION

We consider a cooperative multiple access channel (CMAC), where two sources cooperates to send their independent messages to a common destination. We investigate the diversity order of the CMAC, that one or both messages are received incorrectly. It is common to assume that Gaussian signalling is employed, see e.g. [1]. The mutual information then approaches infinity at high SNR. In practice, the modulated symbols are drawn from a finite alphabet set and so the mutual information is always bounded. Taking into account possible link failures in the handshaking phase that is used to set up cooperation, and also the use of modulation with bounded mutual information, it is not clear under what conditions the maximum (end-to-end) diversity order of two can be achieved.

We focus on a three-phase protocol in the CMAC, based on the decode-and-forward scheme that is similar to [1], see Section II-A. To address the effects of bounded mutual information, Theorem II gives necessary and sufficient conditions to achieve the maximum diversity, for a class of modulation schemes that satisfy some mild conditions given in Section II-A. In related works, only necessary conditions are given for the CMAC with a four-phase protocol [2] and for the multiple access relay channel and relay channel [3].

II. MODEL

A. Protocol

Two sources $S_1$, $S_2$ wish to deliver independent messages to a destination $D$, and may cooperate to do so. To facilitate implementation in practice, a three-phase protocol suitable for half-duplex channels is considered for cooperation, see Fig. 1. Each cycle of the protocol consists of $N$ channel uses. Phase $i \in \{1, 2, 3\}$ lasts $\alpha_i N$ channel uses, where $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

Phase 1: $S_1$ transmits a codeword based on its message $W_1$ at rate $R_1$, $S_2$ tries to decode for $W_1$.

Phase 2: $S_2$ transmits a codeword based on its message $W_2$ at rate $R_2$, $S_1$ tries to decode for $W_2$.

Phase 3, case I: Suppose at least one source decodes incorrectly. Then, both sources keep silent.

Phase 3, case II: Suppose both sources decode correctly. Then, both sources generate a codeword using $W = (W_1, W_2)$ based on the same codebook. Then, they transmit using the Alamouti space-time block code (STBC) [4], in a similar manner as a co-located two-antenna transmitter.

We assume all nodes know if cooperation is possible after Phase 2. We also assume perfect timing synchronization.

B. Channel

Each communication link, from node $i$ to node $j$, is modeled as a complex-valued Gaussian channel $Y_j = H_{ij} X_i + N_i$, where $Y_j$ is the received signal, $H_{ij}$ is the channel gain, $X_i$ is the transmitted symbol and $N_i$ is independent noise (not necessarily Gaussian distributed) with unit variance. The channel is quasi-static in each protocol cycle. We assume a reciprocal channel, i.e., $|H_{12}| = |H_{21}|$. We assume $H_{ij}$ is Rayleigh-distributed and independent for different links.

For simplicity, a common modulation is employed for all links. Thus, the mutual information function is the same for all transmissions. Assuming receiver $j$ performs coherent detection with knowledge of $H_{ij}$, the mutual information can be expressed as $I(\gamma_{ij})$, as a function of the SNR $\gamma_{ij}$.

C. Rate Region for Destination

After Phase 3 ends, $D$ uses all received codewords to jointly decode messages $W_1, W_2$, via typical-set decoding. We obtain the rate region for $(R_1, R_2)$ that ensures reliable decoding.

Consider case I in Phase 3. The received codewords for $W_1, W_2$ are available only in Phases 1, 2, respectively. Thus, $D$ separately decodes the message $W_1, W_2$. The rate region is thus $R_i \leq \alpha_i I_{ij}, i = 1, 2$. Let $E_i$ be the event that $R_i > \alpha_i I_{ij}, i.e.,$ the transmission from node $i$ to node $j$ fails. Thus, the event that $W_1$ or $W_2$ is not decoded correctly is $E_{1D} \cup E_{2D}$.

Consider case II in Phase 3. $D$ uses the received codewords in all phases to jointly decode the messages $W_1, W_2$. By
using independent codebooks in the all transmissions, it can be shown that the rate region is
\[
\begin{align*}
R_1 & \leq \alpha_1 I_{1D} + \alpha_3 I_{STBC}, \\
R_2 & \leq \alpha_2 I_{2D} + \alpha_3 I_{STBC}, \\
R_1 + R_2 & \leq \alpha_1 I_{1D} + \alpha_2 I_{2D} + \alpha_3 I_{STBC}.
\end{align*}
\]
Here, \( I_{STBC} = I(\gamma_{1D} + \gamma_{2D}) \) is the mutual information achieved by the Alamouti STBC scheme, as the channel powers are added. We denote the event that \( R_1, R_2 \) do not satisfy at least one of the inequalities \((1a), (1b), (1c)\) as \( \mathcal{E}_{STBC} \).

## III. DIVERSITY ORDER

We analyze the diversity of the outage event \( O \) that at least one of the messages \( W_1, W_2 \) is not received correctly by \( D \) at the end of the three-phase protocol cycle.

### A. Mutual Information Function

The mutual information function \( I(\cdot) \) depends on the modulation used and the noise distribution at the receiver. We assume \( I(\gamma), \gamma \geq 0, \) that satisfies these properties.

\begin{enumerate}
\item \( I(0) = 0 \) and \( I(\gamma) \) is a monotonically increasing function.
\item \( I(\gamma_1) + I(\gamma_2) > I(\gamma_1 + \gamma_2), \gamma_1 + \gamma_2 > 0 \).
\end{enumerate}

Property P1 allows the inverse \( I^{-1}(R) > 0 \) to exist for \( R > 0 \). Property P2 means that accumulating mutual information is strictly better than accumulating SNR.

In practice, modulated signals are drawn from a finite alphabet set, thus the mutual information is bounded, i.e., \( I^* = \lim_{\gamma \to \infty} I(\gamma) < \infty \). For bounded \( I(\gamma), \gamma \geq 0, \) in addition to P1 and P2, we assume P3 below is also satisfied.

\begin{enumerate}
\item \( I(\gamma) \geq I^*[1 - \exp(-k\gamma)] \) where \( k > 0 \).
\end{enumerate}

Properties P1 to P3 are typically satisfied, see Example 1.

**Example 1:** Assume Gaussian distributed noise \( N \). If Gaussian signaling is used, i.e., \( X \) is Gaussian distributed, then \( I(\gamma) = \log(1 + \gamma) \). If BPSK modulation is used, i.e., \( X \in \{\pm 1\} \) with equal probability, then \( I(\gamma) \) is given in [3, Example 9.12]. For Gaussian signaling \( I^* \) is unbounded, while for BPSK modulation \( I^* = 1 \). It can be shown that P1, P2 hold in both cases, and P3 hold with \( k = 1 \) in the latter case.

### B. Analysis

Let \( \Pr(\mathcal{E}) = \gamma^{-r} \) if \( \lim_{\gamma \to \infty} \frac{\log \Pr(\mathcal{E})}{\log \gamma} = -r \). We say the diversity order is \( r \). The inequalities \( \geq, \leq \) are defined similarly.

From Section II-C we can write the outage probability as
\[
\Pr(O) = \Pr(\mathcal{E}_{12} \cup \mathcal{E}_{21}) \Pr(\mathcal{E}_{1D} \cup \mathcal{E}_{2D} | \mathcal{E}_{12} \cup \mathcal{E}_{21}) \\
+ \Pr(\mathcal{E}_{12} \cup \mathcal{E}_{21}) \Pr(\mathcal{E}_{STBC} | \mathcal{E}_{12} \cup \mathcal{E}_{21})
\]
where \( \mathcal{E}_{12} \cup \mathcal{E}_{21} \) is the event that case I in Phase 2 occurs and the complement \( \mathcal{E}_{12} \cup \mathcal{E}_{21} \) is the event that case II occurs.

**Lemma 1:** Suppose \( I(\cdot) \) satisfies property P1. Then, \( \Pr(O) \approx \Pr(\mathcal{E}_{STBC}) \) if \( R_i < \alpha_i I^*, i = 1, 2 \).

**Proof:** From \((3)\), an upper bound for \( \Pr(O) \) is given by
\[
\begin{align*}
\Pr(O) & \leq \Pr(\mathcal{E}_{12} \cup \mathcal{E}_{21}) \Pr(\mathcal{E}_{1D} \cup \mathcal{E}_{2D}) + \Pr(\mathcal{E}_{STBC}) \\
& \leq [\Pr(\mathcal{E}_{12}) + \Pr(\mathcal{E}_{21})][\Pr(\mathcal{E}_{1D}) + \Pr(\mathcal{E}_{2D})] + \Pr(\mathcal{E}_{STBC}) \approx \gamma^{-2} + \Pr(\mathcal{E}_{STBC}) \approx \Pr(\mathcal{E}_{STBC}).
\end{align*}
\]

### IV. CONCLUSION

We have provided necessary and sufficient conditions to achieve the maximum diversity order for a decode-and-forward scheme. These conditions capture the interplay of the diversity order and the modulation used. These results can be used to guide the design of capacity-achieving codes in practice.

**APPENDIX**

The following lemma will be useful for our proof.

1Nevertheless, to improve the error probability, any source that decodes correctly in case 1 may retransmit in Phase 3.
2A fixed inter-source channel can still reduce the diversity order \( 6 \). Here, all channel power scales similarly, e.g., due to increase in transmission power.
Lemma 2: Suppose $\mathcal{I}(\cdot)$ satisfies property P1. Let $Q = (2r/\bar{\gamma})\gamma$ be chi-square distributed with $2r$ degrees of freedom, so $\mathbb{E}[\gamma] = \bar{\gamma}$. Then, for $R > 0, \alpha > 0$,

$$
\Pr(E) = \Pr(R > \alpha\bar{I}(\gamma)) = \left\{ \begin{array}{ll}
\bar{\gamma}^{-r}, & R < \alpha\bar{I}^*; \\
1, & \text{otherwise}
\end{array} \right.
$$

(6)

Proof: If $R > \alpha\bar{I}^*$, clearly $R > \alpha\bar{I}(\gamma)$ with probability one. If $R < \alpha\bar{I}^*$, $\bar{I} = \bar{I}^{-1}(R/\alpha) > 0$ is well defined due to P1. Thus, $\Pr(E) = \Pr(Q < (2r/\bar{\gamma})\gamma) = \bar{\gamma}^{-r}$, as $\Pr(Q < x)$ is given by the regularized Gamma function $\mathcal{P}(x/2)$ which approaches $kx^r$ for small $x$, where $k$ is a positive constant.

For Rayleigh-fading channels, $\gamma_{12}, \gamma_{1D}, \gamma_{2D}$ are chi-square distributed with $r = 1$, while $\gamma_{1D}$ and $\gamma_{2D}$ is chi-square distributed with $r = 2$. From (6), the diversity order is one for $\Pr(E_{12}), \Pr(E_{1D}), \Pr(E_{2D})$, assuming the corresponding rate conditions of the form $R < \alpha\bar{I}^*$ are satisfied. Also, since $\bar{I}_{\text{STBC}} = \bar{I}_{\gamma_{1D}} + \bar{I}_{\gamma_{2D}}$, we get $\Pr(R > \alpha\bar{I}_{\text{STBC}}) = \bar{\gamma}^{-2}$ if $R < \alpha\bar{I}^*$. These arguments will be used frequently.

We now prove Theorem 1 specifically that $5a$ are all necessary and sufficient to achieve the diversity order of two.

Proof of Theorem 1: Suppose $5a$ holds. From Lemma 2, we only need to consider the diversity order of $\Pr(E_{\text{STBC}})$. Let $E_1, E_2, E_3$ be the error events that (11), (12), (13) are not satisfied, respectively. Since $E_{\text{STBC}} = \{E_1 \cup E_2 \cup E_3\}$, we can bound $\Pr(E_{\text{STBC}})$ as

$$
\max\{\Pr(E_1), \Pr(E_2), \Pr(E_3)\} \leq \Pr(E_{\text{STBC}}) \leq \Pr(E_1) + \Pr(E_2) + \Pr(E_3).
$$

By definition, the diversity is $r_i$ if $\Pr(E_i) = \bar{\gamma}^{-r_i}$. Thus,

$$
\bar{\gamma}^{-\min\{r_1, r_2, r_3\}} \leq \Pr(E_{\text{STBC}}) \leq \bar{\gamma}^{-\min\{r_1, r_2, r_3\}}.
$$

(7)

That is, $\min\{r_1, r_2, r_3\}$ exactly determines the overall diversity order. Given that $5a$ holds, it can be shown that

$$
\left\{ \begin{array}{l}
2, \quad r_1 = \alpha_3\bar{I}^*; \\
1, \quad \text{otherwise}
\end{array} \right.
$$

(8)

holds for $i = 1, 2$ and also that

$$
\left\{ \begin{array}{l}
2, \quad R_1 + R_2 \leq (\min\{\alpha_1, \alpha_2\} + \alpha_3)^*; \\
1, \quad \text{otherwise}
\end{array} \right.
$$

(9)

holds; the proofs are given in the next two subsections. It follows straightforwardly from (7), (8), (9) that $5a, 5c$ are both necessary and sufficient conditions. We have shown that $5b, 5c$ are necessary and sufficient conditions to achieve the maximum diversity order assuming $5a$ holds. From Remark 1, $5a$ is also a necessary condition. Thus, all $5a, 5b, 5c$ are necessary and sufficient conditions.

1) Proof of (5b): We prove (8) for $i = 1$; the case for $i = 2$ is similar. First, consider $R_1 \leq \alpha_3\bar{I}^*$. Given $\gamma_{1D} > 0$,

$$
\Pr(E_1|\gamma_{1D}) = \Pr(\tau > I_{\text{STBC}}|\gamma_{1D})
$$

(10)

with $\tau \triangleq (R_1 - \alpha_3\bar{I}_{1D}/\alpha_3$. If $R_1 \leq \alpha_3\bar{I}_{1D}$, $\Pr(E_1|\gamma_{1D}) = 0$. Consider $R_1 > \alpha_3\bar{I}_{1D}$, i.e., $\gamma_{1D} < \bar{I}^{-1}(R_1/\alpha_3) \triangleq \gamma^*$. Then

$$
\begin{align}
\Pr(E_1|\gamma_{1D})^2 \leq & \Pr(\tau > I_{2D}|\gamma_{1D}) \leq 1 - \exp(-\bar{I}^{-1}(\tau/\bar{\gamma})) \\
& \leq \frac{\bar{\gamma}}{\bar{\gamma}} \leq \frac{\bar{\gamma}}{\bar{\gamma}} \leq \log(I_{2D}/\bar{I}_{1D}) + \epsilon
\end{align}
$$

(11)

where $\epsilon = \log(\alpha_3/\alpha_1)$. Here, (a) is due to $I_{\text{STBC}} \geq I_{2D}$, (b) is due to $\gamma_{2D}$ being exponential distributed, (c) is due to $\exp(-x) \geq 1 - x$ for $x \geq 0$, (d) is due to P1, P3 and (e) is due to $\tau \leq \bar{I}^{-1} - \alpha_3/\alpha_3\bar{I}_{1D}$. Thus,

$$
\Pr(E_1) = \int_{0}^{\gamma_{1D}} \Pr(E_1|\gamma_{1D})d\gamma_{1D} \leq \frac{I_1 + I_2}{\bar{\gamma}}
$$

(12)

Here, $\bar{I}_{2D}(z) = -\int_{0}^{\gamma_{1D}} \log(1 - x/\bar{\gamma})dx$ is the digilogarithm function, or alternatively $\bar{I}_{2D}(z) = \sum_{n=1}^{\infty} z^n/n^2$ which is convergent for $|z| \leq 1$ [7]. Moreover, we get $I_2 = \epsilon(1 - \exp(-\bar{\gamma}^*/\bar{\gamma})) \approx \epsilon$. Thus, $\Pr(E_1) \approx \epsilon^{-2}$ if $R_1 \leq \alpha_3\bar{I}^*$.

Next, consider $R_1 > \alpha_3\bar{I}^*$. Since $I_{\text{STBC}} < I^*$, a lower bound for $\Pr(E_1)$ is $\Pr(E_1) > \Pr(R_1 - \alpha_3\bar{I}^* > \alpha_1\bar{I}_{1D})$. Since $R_1 - \alpha_3\bar{I}^*$ is positive, we can apply Lemma 2 to get

$$
\Pr(R_1 - \alpha_3\bar{I}^* > \alpha_1\bar{I}_{1D}) \approx \bar{\gamma}^{-1}, R_1 < (\alpha_1 + \alpha_3)\bar{I}^*, \quad \text{otherwise}
$$

Here, (13) is due to $\alpha_1 \leq \alpha_2$. We have

$$
\Pr(E_3) = \Pr(R_1 + R_2 > (\alpha_1 + \alpha_3)\bar{I}_{1D}) \approx \bar{\gamma}^{-2}
$$

if $R_1 + R_2 > (\alpha_1 + \alpha_3)\bar{I}^*$. Thus, $r_3 = 2$ if $R_1 + R_2 \leq (\alpha_1 + \alpha_3)\bar{I}^*$.

Next, consider $R_1 + R_2 > (\alpha_1 + \alpha_3)\bar{I}^*$. Since $I_{\text{STBC}} < I^*$ and $I_{1D} < I^*$, a lower bound for $\Pr(E_3)$ is

$$
\Pr(E_3) > \Pr(R_1 + R_2 - (\alpha_1 + \alpha_3)\bar{I}^* > \alpha_2\bar{I}_{2D})
$$

Since $R_1 + R_2 - (\alpha_1 + \alpha_3)\bar{I}^*$ is positive, we can apply Lemma 2 to get $r_3 \leq 1$, in a similar manner as in (12).

The proof for the case $\alpha_2 \leq \alpha_1$ is similar.

REFERENCES


