Data Offloading in Load Coupled Networks: Solution Characterization and Convexity Analysis

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Abstract—We provide a general framework for data offloading in a cellular network, where some demand of users is served in a complementary network. The complementary network is either a small-cell network that possibly shares the same resources as the cellular network, or a WiFi network that uses orthogonal resources. For a given demand served in a cellular network, the load, or the level of resource usage, of each cell depends in a non-linear manner on the load of other cells due to the mutual coupling of interference seen by one other. Taking into account this load-coupling effect, we choose the users’ demand to be served in the cellular network or the complementary network, so as to maximize a utility function. We establish conditions for which the optimization problem has a feasible solution and is convex, and hence tractable to numerical computations.

Index Terms—Data offloading, load coupling, small-cell network, WiFi network, feasibility, convexity.

I. INTRODUCTION

Fueled by multimedia mobile applications, the demand for mobile data is rising rapidly. Data traffic is also projected to grow at a compound annual growth rate of 78% from 2011 to 2016 [1]. In practice, cellular networks and the conventional infrastructure cannot grow as fast to match the increase in demand. One promising solution currently considered by cellular operators is to employ data offloading, also known as mobile cellular traffic offloading [2], [3]. In data offloading, the data of cellular users is intentionally delivered by complementary networks, namely small cells such as Picocells and Femtocells, or WiFi networks. This reduces the data demand on the regular cellular networks and hence eases traffic congestion.

In a cellular network, frequency reuse is employed, and thus base stations using the same frequency band interfere with one another. We refer to the average level of channel resource usage in a cell as its load. Due to the mutual coupling of the interference and the requirement to serve a specific demand for each cell, the load of a cell depends on the load of other cells. This led to a non-linear coupling relation of the cells’ loads, making the analytical characterization of the load challenging. Recently, an analytical signal-to-interference-and-noise-ratio (SINR) model that takes into account the load of each cell is employed [4], [5], resulting in a non-linear load coupling equation for which analytical results are obtained in [6].

In this paper, we consider two separate scenarios in which the cellular network offloads to a complementary network. The complementary network is either a small cell or a WiFi network. In our analysis, the small-cell network shares the resource with the cellular network. For WiFi offloading, the resource used in the WiFi network is orthogonal to that of the cellular network. The advantage of serving users in a particular network is measured by three representative types of utility functions, that differ in their degrees of accounting for user fairness. To model the inter-dependency of the load, we employ the load coupling equation in [4]–[7].

Our contributions are as follows. Based on a unified framework for the problem of data offloading, we characterize a solution of the load-coupling equation. We formulate a utility-maximization problem in which the users’ demand can be served in either the (regular) cellular network or the (small cell or WiFi) complementary network. We establish conditions for which the optimization problem has a feasible solution and is convex, and hence tractable to numerical computations. The main tool we employ for analysis is based on the Perron-Frobenius theorem and other related results [8].

Section II gives the system model of the load-coupled network. Section III obtains the feasibility condition for the load. The data offloading problem is formulated with convexity analysis given in Section IV. Section V concludes the paper.

Notations: We denote a positive matrix as $A > 0$ if $a_{ij} > 0$ for all $i, j$. Similarly, we denote a non-negative matrix as $A \geq 0$ if $a_{ij} \geq 0$ for all $i, j$. We refer to a positive/non-negative matrix with the inequality sign reversed, in a similar manner. Vectors are denoted similarly.

II. SYSTEM MODEL

We consider a cellular network consisting of $n$ base stations that can interfere with each other. We focus on the downlink communication scenarios where base station $i \in \mathcal{N} \triangleq \{1, \cdots , n\}$ transmits with power $p_i > 0$. We refer to cell $i$ interchangeably with base station $i$. For notational convenience, we collect all power $\{p_i\}$ as vector $p > 0$.

Each base station $i$ serves one distinct group of users in set $\mathcal{J}_i$, $|\mathcal{J}_i| \geq 1$. User $j \in \mathcal{J}_i$ is served in cell $i$ up to a maximum rate of $D_{ij}$, expressed in bit instead of bit for convenience. Thus, the data can be interpreted as best-effort data to be served as much as possible subject to network conditions.

A. Data Offloading to Alternative Network

We shall consider data offloading, where the demand $d_{ij}$ is served in the cellular network, while the demand $d'_{ij}$ is
offloaded to be served in a complementary network. We take \(d_{ij}\) and \(d'_{ij}\) as variables to be optimized, subject to the constraint \(d_{ij} + d'_{ij} \leq D_{ij}\). We collect all demands \(\{d_{ij}, j \in J_i, i \in N\}\) as vector \(\mathbf{d} \geq 0\); similarly \(\{d'_{ij}\}\) is collected as \(\mathbf{d}' \geq 0\). In general not all users are served in the regular cellular network.

We assume there is at least one user \(j\) in cell \(i\) with \(d_{ij} > 0\), otherwise \(p_i = 0\) and so base station \(i\) can be omitted. Similarly, we assume that there is at least one user \(j\) in cell \(i\) with \(d'_{ij} > 0\), i.e., the complementary cell is not redundant. We assume the complementary network for offloading is densely deployed to offload as much as network conditions allows.

For exposure, we assume that base station \(i\) offloads to only one complementary cell. The generalization where the base station can offload to more than one small cell is straightforward.

We consider two types of complementary network, either a small cell or a WiFi cell. For small-cell offloading, both the regular cellular network and small-cell network use the same frequency band; hence the two networks interfere with one another. For WiFi offloading, the frequency band used in the WiFi network is orthogonal to that of the cellular network; hence the two networks do not interfere with each other.

**B. Load Coupling Model**

We first consider the load coupling model for the cellular network. The extension to the case with a complementary network is given in Remark 1 and Remark 2 later. Let \(\mathbf{x} = [x_1, \ldots, x_n] \geq 0\) be the load of the cellular network, where \(x_i\) measures the fractional usage of resource in generic cell \(i\). In LTE systems, the load can be interpreted as the expected fraction of the time-frequency resources that are scheduled to deliver data. We model the SINR of user \(j\) in cell \(i\) as [4]–[7]

\[
\text{SINR}_{ij}(\mathbf{x}) = \frac{p_i g_{ij}}{\sum_{k \in N \setminus \{i\}} p_k g_{kj} x_k + \sigma^2} \tag{1}
\]

where \(\sigma^2\) represents the noise power and \(g_{ij}\) is the gain of the channel from base station \(i\) to user \(j\); here \(g_{kj}\) represents the channel gain from the interfering base station. The SINR model (1) gives good approximation for more complicated cellular models [7]. Intuitively, \(x_k\) can be interpreted as the probability of receiving interference from cell \(k\) on all the sub-carriers of the resource unit. Thus, the term \((p_k g_{kj} x_k)\) is interpreted as the expected interference with expectation taken over time, frequency for all transmission.

Since Gaussian-signalling is the worst-case noise distribution for mutual information [9], an achievable rate is given by \(r_{ij} = B \log(1 + \text{SINR}_{ij})\) nat/s, where \(B\) is the bandwidth and \(\log\) is the natural logarithm. To deliver a demand of \(d_{ij}\) for user \(j\), the \(i\)th base station thus uses \(x_{ij} \triangleq d_{ij} / r_{ij}\) resource units. We assume that \(K\) resource units are available. Summing the resource units over all users in cell \(i\), we get the load for the cell \(x_i = \sum_{j \in J_i} x_{ij} / K\), i.e.,

\[
x_i = \frac{1}{KB} \sum_{j \in J_i} \log \left(1 + \text{SINR}_{ij}(\mathbf{x})\right) \triangleq f_i(\mathbf{x}) \tag{2}
\]

for \(i \in N\). We normalize \(d_{ij}\) by \(KB\) in (2) and so we let \(KB = 1\) without loss of generality. In vector form, we have

\[
x = f(\mathbf{x}; \mathbf{d}, \mathbf{p}) \tag{3}
\]

where \(f(\mathbf{x}; \mathbf{d}, \mathbf{p})\) collects the elements \(f_i(\mathbf{x}), i \in N\), with dependence on \(\mathbf{d}\) and \(\mathbf{p}\) made explicit. We call (3) the non-linear load coupling equation, as the load \(\mathbf{x}\) is not readily solved in closed-form. For a given \(\mathbf{d}, \mathbf{p}\), a fixed-point solution \(\mathbf{x} \geq 0\) that satisfies the non-linear load coupling equation is said to be feasible. In practice, the load is subject to the constraint \(\mathbf{x} \leq 1\) to reflect the limitation on the given resources. In this paper, the constraint is not explicitly considered in the analysis. Incorporating the upper bound on \(\mathbf{x}\) will be a future extension of the current work.

**Remark 1**: For the cellular network with a small-cell network, the two networks operate in the same frequency band and can be treated as one integrated network. Thus the load coupling equation holds with the following two modifications. First, the set of base stations includes also those in the small cells. Second, each user can be split into two virtual users: one with demand \(d_{ij}\) served in the regular cell and another with demand \(d'_{ij}\) served in the corresponding small cell.

**Remark 2**: For the cellular network with a WiFi network, the two networks operate in different frequency bands. We assume the WiFi network also submits to the load-coupling system relation. So the load coupling equation also holds separately for the WiFi network by replacing \(\mathbf{x}, \mathbf{d}, \mathbf{p}\) with the corresponding WiFi quantities denoted by \(\mathbf{x}', \mathbf{d}', \mathbf{p}'\).

Regardless of whether the complementary network is a small cell or WiFi network, the allocation of \(\{d_{ij}, d'_{ij}\}\) is not decoupled due to the constraint \(d_{ij} + d'_{ij} \leq D_{ij}\).

### III. Feasibility of Load Solution

We explore a fundamental property of the non-linear load coupling equation, namely, feasibility of the load solution. For clarity, we consider the regular cellular network without any complementary network; the results extend straightforwardly to the case with complementary network via Remarks 1, 2.

Lemma 1 states that feasibility can be checked by a simpler problem, specifically via the linear load coupling equation (4).

**Lemma 1**: Given \(\mathbf{d}\) and \(\mathbf{p}\), a feasible load \(\mathbf{x} \geq 0\) exists in (3) if and only if a solution \(\mathbf{x} \geq 0\) exists in the following linear load coupling equation

\[
x = \mathbf{H}(\mathbf{d}, \mathbf{p}) \cdot \mathbf{x} + \mathbf{c}(\mathbf{d}, \mathbf{p}). \tag{4}
\]

Here, \(\mathbf{c}(\mathbf{d}, \mathbf{p}) \triangleq f(0_n; \mathbf{d}, \mathbf{p})\), where \(0_n\) is the length-\(n\) all-zero vector, and \(\mathbf{H}(\mathbf{d}, \mathbf{p}) \geq 0\) is the real matrix with \((i,k)\)th element

\[
h_{ik} = \begin{cases} 
0, & \text{if } i = k; \\
\left(\frac{p_i}{p_k}\right) \sum_{j \in J_i} g_{kj} d_{ij} / g_{ki}, & \text{if } i \neq k
\end{cases} \tag{5}
\]

for \(1 \leq i \leq n\) and \(1 \leq k \leq n\). Note that \(\mathbf{c}(\mathbf{d}, \mathbf{p}) > 0\) because at least one \(d_{ij}\) in cell \(i\) is positive.

**Proof**: From Theorem 8 and Theorem 11 in [6].

Next, we treat \(\mathbf{d}\) and \(\mathbf{p}\) as variables to be optimized, so as to study how they affect the feasibility of the load. Our main
result is stated in Theorem 1 below, which gives the necessary and sufficient condition for a feasible $x$ to exist.

We make some preparation before stating the theorem. Let $\Lambda(d) \geq 0$ be the $n$-by-$n$ real matrix with the $(i, k)$th element
\begin{equation}
\lambda_{ik} = \begin{cases}0, & \text{if } i = k; \\ \sum_{j \in J} g_{kj}d_{ij}/g_{ij}, & \text{if } i \neq k\end{cases}
\end{equation}
for $1 \leq i \leq n$ and $1 \leq k \leq n$. We can therefore express the matrix $H(d, p)$ in (4) as
\begin{equation}
H(d, p) = \text{diag}(p) \cdot \Lambda(d) \cdot \text{diag}(p)^{-1}
\end{equation}
where $\text{diag}(p)$ denotes the diagonal matrix with diagonal elements $p$. The effects of $p$ and $d$ are thus decoupled into three matrices, and so (4) becomes
\begin{equation}
\tilde{x} = \Lambda(d)x + \tilde{c}(p, d)
\end{equation}
where $\tilde{x} \triangleq \text{diag}(p)^{-1}x$ and $\tilde{c}(d, p) \triangleq \text{diag}(p)^{-1}c(d, p)$.

**Theorem 1:** Given $p > 0$ and $d \geq 0$, a feasible $x$ in (3) exists if and only if
\begin{equation}
r(\Lambda(d)) < 1
\end{equation}
where $r(\Lambda)$ is the spectral radius of matrix $\Lambda$, given by the absolute value of the largest eigenvalue of $\Lambda$.

**Proof:** By Lemma 1, it is sufficient to consider the linear load coupling equation (4). Since $p > 0$, every base station $i$ serves some positive demand and so $\sum_{j \in J} d_{ij} > 0$. Thus, $\Lambda(d) \geq 0$ and $c(d, p) > 0$. Hence, we can apply the Perron-Frobenius theorem in [8, Theorem 5.1] to conclude that (9) is necessary and sufficient for a feasible $\tilde{x}$ to exist in (8).

**Theorem 2:** Suppose a feasible load does not exist for a given demand $d \geq 0$ and power $p > 0$. Then no feasible load can exist by changing only $p$. However, a feasible load can always exist by reducing $d$.

**Proof:** The spectral radius $r(\Lambda(d))$ depends only on the demand $d$. Hence, changing the power $p$ does not affect the existence of the feasible load. But scaling the demand vector uniformly by a positive factor $\kappa$ allows the spectral radius to be scaled also by $\kappa$. Hence the spectral radius can always be made smaller than one by reducing the demand such that a feasible load exists. 

**IV. DEMAND OFFLOADING**

We model the benefit of serving the demand in a network via representative utility functions. Under the framework of maximizing the sum utility, we investigate the convexity of the solution space, which hence affects the difficulty of numerical computations of the optimal solution.

**A. Utility for Maximization**

We assume that demand $d_{ij}$ of user $j$ is served in the $i$th regular cell, while $d_{i}'_{ij}$ of the remaining demand is served in the corresponding complementary cell. Our objective is to maximize the sum utility
\begin{equation}
U^{\text{sum}}(\hat{d}, \hat{d}') \triangleq \sum_{i \in N} \sum_{j \in J} k_{ij}U(d_{ij}) + k_{ij}'U(d_{i}'_{ij})
\end{equation}
where $U(d)$ is the utility function. The weights $k_{ij}$ and $k_{ij}'$ take into account the priority of the user and the network. The utility function quantifies the value of serving the demand to the cellular operator or user in terms of, for instance, the revenue collected from the access service, and the fairness of serving the demand to multiple users.

To give insights, $U(d)$ is chosen to be the following representative functions, namely the linear (LIN), logarithmic (LOG), and double-logarithmic (DLOG) utility functions:
\begin{align}
\text{LIN} & : U(d) = d, \\
\text{LOG} & : U(d) = \log(d), \\
\text{DLOG} & : U(d) = \log(\log(1 + d)).
\end{align}
The utility functions are monotonically increasing and hence one-to-one functions. The LIN utility models the scenario where serving an additional demand unit results in an additional unit of utility. For LOG utility, serving an additional demand unit to a user with a low demand results in more utility. Intuitively, this results in a fairer demand distribution among users but could result in a smaller revenue to the operator as less total demand is served. Thus, the LOG utility trades revenue maximization with user fairness. The DLOG utility further emphasize fairness, because it favours low-demand users even more. We note that the last two utility functions would not assign zero demand to any user, because the sum utility is then negative infinity. The generalization to a class of functions is considered in Remark 3 later.

For exposure, we make the same-demand assumption that for each cell $i$, user $j \in J_i$ is served the same demand $d_{ij} = d_i$. The demand of the corresponding cell in the complementary network is subject to the same assumption, i.e., $d_{i}'_{ij} \equiv d_i'$. In effect, we focus on varying the cell-level demand vectors $d = [d_1, \ldots, d_n]^T$ and $d' = [d_1', \ldots, d_n']^T$, $d_i + d_i' \leq D_i, \forall i \in N$. To ensure all cells are active, we impose $d > 0$ and $d' > 0$.

**B. Optimization Problem**

1) WiFi as Alternative Network: We first formulate the optimization problem with WiFi as the complementary network. Mathematically, our data offloading problem is
\begin{align}
(P0) \quad \max_{\hat{d}, \hat{d}'} & \sum_{i \in N} \sum_{j \in J_i} k_{ij}U(d_{ij}) + k_{ij}'U(d_{i}'_{ij}) \quad (1a) \\
\text{s.t.} & \quad \hat{d} \in F \triangleq \{\hat{d} : r(\Lambda(\hat{d})) < 1\} \quad (1b) \\
& \quad \hat{d}' \in F' \triangleq \{\hat{d}' : r(\Lambda'(\hat{d}')) < 1\} \quad (1c) \\
& \quad d_i + d_i' \leq D_i, \forall i \in N \quad (1d)
\end{align}
where $\Lambda, \Lambda'$ corresponds to (6) for the regular cellular network and the WiFi network, respectively. The constraints (1b), (1c) follow from Theorem 1 and Remark 2; we call $F$ and $F'$ the feasibility sets. The last constraint ensures that the total demand is not more than the demand $D_i$ requested by the users; this is because, practically, serving more than $D_i$ wastes resources and is not beneficial to the cellular operator. A solution always exists because we can always reduce the
demand such that the spectral radius is less than one to satisfy constraints (14b) and (14c) (see the proof of Theorem 2 for details) and simultaneously to satisfy constraint (14c).

For convenience, we introduce the transformation $y_i = U(d_i)$ for $i \in \mathcal{N}$ and denote $d = [U^{-1}(y_1), \cdots, U^{-1}(y_n)]^T \triangleq g(y)$. Let $k_i \triangleq \sum_{j \in \mathcal{J}_i} k_{ij}$. We make similar definitions for $y_i'$ and $k_i'$ corresponding to the complementary cells. Our transformed data offloading problem is then

$$
(P1) \max_{y, y' \in \mathcal{Y}} \sum_{i \in \mathcal{N}} k_i y_i + k_i' y_i' \quad (15a)
$$

s.t. $y \in \mathcal{F} \triangleq \{y : r(\Lambda(g(y))) < 1\}$ \hspace{1cm} (15b)

$$
y' \in \mathcal{F}' \triangleq \{y' : r(\Lambda'(g(y'))) < 1\} \quad (15c)
$$

$$
U^{-1}(y_i) + U^{-1}(y_i') \leq D_i, \forall i \in \mathcal{N} \quad (15d)
$$

where $\mathcal{Y} = \mathbb{R}_n^m$ for LIN utility and $\mathcal{Y} = \mathbb{R}^n$ for LOG and DLOG utilities. Here $\mathcal{F}$ (and similarly $\mathcal{F}'$) is the transformed feasibility set with complement $\mathcal{F}^c = \mathbb{R}^n \setminus \mathcal{F}$. For LIN utility, Problem P1 is the same as Problem P0, and thus $\mathcal{F} = \mathcal{F}'$. We note that the objective function is always linear. For any of the three utility functions, it can be checked that the set of $y, y'$ subject only to (15d) is convex. Now if $\mathcal{F}$ (and similarly $\mathcal{F}'$) is a convex set, $P1$ is a convex optimization problem for which numerically efficient solvers exist [10]. In summary, it is sufficient to obtain conditions for which $\mathcal{F}$ is convex, in order to ascertain if problem P1 is convex.

2) Small-cell as Alternative Network: The optimization problem is similar to problem P0, except that the feasibility sets in (14b) and (14c) are merged as a single feasibility set subject to $r(\Lambda''(d, d')) < 1$ where $\Lambda''$ includes the base stations in both the regular cells and the small cells, see Remark 1. By similar arguments as before, for the transformed data offloading problem to be convex, it suffices to check if $\mathcal{F}$ that corresponds to $\Lambda''(d, d')$ is convex.

C. Two Base Stations

To gain some understanding for the convexity of $\mathcal{F}$, let us study the case of $n = 2$ base stations. The spectral radius can be obtained in closed-form, which allows the (non-transformed) feasibility set $\mathcal{F}$ to be given by all $d = [d_1, d_2]^T$ that satisfies

$$
d_1 d_2 \left( \sum_{j \in \mathcal{J}_1} g_{1j} \right) \left( \sum_{j \in \mathcal{J}_2} g_{2j} \right) < 1. \quad (16)
$$

Thus, the feasibility set depends on the channel gains in a non-linear manner. We note that the optimal $(d_1, d_2)$ lies on the inner boundary of $\mathcal{F}$, since to maximize the objective function we must choose $d_1$ or $d_2$ as large as possible. Moreover, the following observations can be made for the utility functions.

Consider LIN utility. The transformed feasibility set $\mathcal{F} = \mathcal{F}'$ is unchanged and, together with the constraint (14c), is plotted in Fig. 1(a). To maximize the sum utility, clearly the optimal solution is to assign either $d_1^* = D_1$ or $d_2^* = D_2$, i.e., an extreme solution. Moreover, the optimal solution is unique.

Consider LOG utility. The feasible solution space in the transformed domain is linear, as shown in Fig. 1(b). To maximize the objective function $k_1 y_1 + k_2 y_2$, an extreme solution is optimal, similar to the LIN utility case. However, the optimal solution is not necessarily unique.

Consider DLOG utility. The feasible solution space in the transformed domain is strictly convex, see Fig. 1(c). The optimal solution is not necessarily an extreme solution, but is always unique. Thus, neither of the demands is likely to be reduced much from its maximum value, as compared to the LIN utility case where one demand is served at maximum value while the other demand is consequently reduced significantly. This suggest that DLOG utility is fairest in data offloading.

In the next section, we shall use more sophisticated analytical tools to shed further insight on the convexity of the transformed feasibility set $\mathcal{F}$ for any $n$.

D. Arbitrary Number of Users

For larger $n$, the spectral radius cannot be computed in closed-form, and it is expected that the dependence on the channel gains remains non-linear and complicated. Nevertheless, an efficient numerical approach is warranted for arbitrary number of base stations $n$. Theorem 3 states the convexity of the feasibility set $\mathcal{F}$ or its complement $\mathcal{F}^c$.

Theorem 3: The following convexity results hold.

- LIN utility: $\mathcal{F}^c$ is convex for $n = 2$. But $\mathcal{F}^c$ is generally not convex (nor concave) for $n \geq 3$.
- LOG utility: $\mathcal{F}$ is convex for $n = 2$. Moreover, $\mathcal{F}$ is strictly convex for $n \geq 3$.
- DLOG utility: $\mathcal{F}$ is strictly convex for any $n \geq 2$.

Proof: The proof for $n = 2$ for all cases was given in Section IV-C. We now consider the case $n \geq 3$ by applying the results in [8], which are closely related to the well-known Perron-Frobenius theorem. First, note that we can express

$$
\Lambda(g(y)) = \text{diag}(g(y_1), \cdots, g(y_n)) \widetilde{\Lambda} \quad (17)
$$

where $g(y_i)$ is the $i$th element of $g(y)$ and the $(i,k)$th element of $\widetilde{\Lambda}$ is

$$
\lambda_{ik} = \begin{cases} 0, & \text{if } i = k; \\ \sum_{j \in \mathcal{J}_i} g_{kj} / g_{ji}, & \text{if } i \neq k. \end{cases} \quad (18)
$$

Consider LIN utility, where $g(y) = y$. Applying [8, Theorem 1.60] known as the linear mapping case to (17), we obtain that $\mathcal{F}^c$ is in general not convex.
Consider LOG utility, where $g(y) = \exp(y)$. The matrix structure in (17) known as the exponential mapping case in [8]. Moreover, $A$ and $AA^T$ are irreducible; see Lemma 2 with definition of irreducibility in the Appendix. These two conditions allow us to apply [8, Theorem 1.63] to show that $\mathcal{F}$ is strictly convex for $n \geq 3$.

Consider DLOG utility, where $g(y) = \exp(\exp(y)) - 1$. The following inequality holds after some calculus and algebraic manipulations:

$$
\begin{align*}
    dU''(d) + U'(d) &= U''(d) \left(1 - \frac{d}{1 + d} \left(1 + \frac{1}{\log(1 + d)} \right) \right) \\
    &< U'(d) \left(1 - \frac{d}{1 + d} \left(1 + \frac{1}{d} \right) \right) = 0 \quad (19)
\end{align*}
$$

where the above inequality is due to $\log(1 + d) < d$ for $d > 0$. From Lemma 3 in Appendix with $x$ and $f(x)$ replaced by $d$ and $U(d)$, respectively, the inverse of $U(d)$, i.e., $g(y)$, is strictly log-convex. Since all diagonal elements of $\text{diag}(g(y))$ are strictly log-convex, by [8, Corollary 1.46], it follows that $\mathcal{F}$ is strictly convex.

**Remark 3:** Theorem 3 applies for a more general class of utility function $U(d)$, namely those that satisfy $dU''(d) + U'(d) < 0$ for $n \geq 3$; note that DLOG is a special case. This conclusion follows immediately from the proof for Theorem 3, in which Lemma 3 was used to show that $g(y)$ is strictly log-convex. Moreover, it follows that $g(y)$ is convex and so the constraint (15d) is convex since the LHS is a sum of convex functions. Hence the optimization problem $P1$ is a convex optimization for this general class of utility functions.

**V. Conclusion**

We have presented a utility-based optimization framework for data offloading in cellular networks, taking into account the inherent coupling relation between the cells. Within this framework, the load-coupling systems feasibility has been studied with respect to power and demand. For offloading, three utility functions that differ in the emphasis on fairness have been considered, and fundamental insights of convexity analysis of the resulting optimization problem have been developed. Our analysis shows that optimal offloading is tractable when proportional fairness is stressed. Moreover, the modeling work provides a structured view on the offloading problem, and our analysis serves as a theoretical reference for empirical simulations and performance evaluation in our future work.

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**Appendix**

For a non-negative matrix $A \in \mathbb{R}^{n \times n}_+$ with elements $\{\lambda_{ij}\}$, its incidence matrix $A \in \{0, 1\}^{n \times n}$ has $(i, j)$th element $a_{ij} = 1$ if $\lambda_{ij} > 0$ and $a_{ij} = 0$ if $\lambda_{ij} = 0$. Denote the $m$-time self multiplication of $A$ as $A^{(m)}$ with element $a_{ij}^{(m)}$. A matrix $A$ is irreducible if its incidence matrix $A$ satisfy $a_{ij}^{(m)} > 0$ for all $i, j$ for some $m \geq 1$.

**Lemma 2:** The matrix $A \in \mathbb{R}^{n \times n}_+$, $n \geq 3$, with $(i, k)$th element given by (18) is irreducible. Also, $(AA^T)$ is irreducible.

**Proof:** From (18), the diagonal elements of $A$ are zeros, while the off-diagonal elements are strictly positive since the channel gains $(g_{ij})$ are positive. Thus, the incidence matrix of $A$ is $A = \mathbb{I}_n - \mathbb{I}_n$, where $\mathbb{I}_n$ is the all-one vector of length $n$ and $\mathbb{I}_n$ is the $n$-by-$n$ identity matrix. Thus $A^2 = (n-2)\mathbb{I}_n + \mathbb{I}_n + \mathbb{I}_n$. Clearly $A^2 > 0$, and so $A$ is irreducible. Similarly, $(AA^T)$ with incidence matrix $A^2$ is irreducible.

**Lemma 3:** Assume $f(x)$ is an increasing function with inverse $g(y) = f^{-1}(y)$. Assume $f(x)$ and $g(y)$ are differentiable. Then $g(y)$ is strictly log-convex if and only if (iff)

$$
xf''(x) + f'(x) < 0. \quad (20)
$$

**Proof:** The second derivative of $\log(g(y))$ is given by $g''(y)/g(y) - (g'(y)/g(y))^2$, so $g(y)$ is strictly log-convex iff

$$
g''(y)/g(y) - (g'(y)/g(y))^2 > 0. \quad (21)
$$

To complete the proof, we shall show that (21) holds. We can write $g(f(x)) = x$. Differentiating with respect to $x$, we get $g'(f(x)) = f'(x)$. Differentiating again with respect to $x$, we get $g''(f(x)) = -f''(x)/(f'(x))^3$. Thus the left-hand side of (21) can be written as $g''(f(x))/g(f(x)) - (g'(f(x)))^2 = -x f''(x)/(f'(x))^3 - 1/(f'(x))^2 = -(x f''(x) + f''(x))/f'(x))$. Since $f'(x) > 0$, (21) holds iff (20), which completes the proof.

**References**


