On the Available Receiver Side Information in Wireless Network Coding

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Abstract—In this paper, we analyze how different amount of receiver side information (RSI) available at the receivers of a broadcast channel will affect the overall network throughput. Two transmission schemes, namely random linear network coding (RLNC) and round robin scheduling (RRS), are considered. Taking into account the differing amount of RSI available at each receiver, closed-form expressions of network throughput and asymptotic throughput at high SNR are obtained for a broadcast channel. Our results show that the asymptotic throughput of RLNC is always no worse than RRS no matter how much RSI is available at each receiver. With a sum constraint for the overall arrival rate of receiver side information in the broadcast channel, we show that the maximum throughput is achieved for RLNC if each receiver accumulates the same amount of RSI.

Index Terms—Broadcast channel, wireless network coding, receiver side information (RSI).

I. INTRODUCTION

As a canonical subnetwork, a broadcast channel where network coding [1] is performed has attracted great interest in research communities [2]-[7]. Most current works simply assume that each receiver knows a priori a certain subset of the transmitted packets as receiver side information (RSI) such that fewer transmissions are required for the receivers to decode all transmitted packets, thus enhancing the network throughput.

For example, in Fig. 1(a), a typical network scenario is shown where a transmitter T has \( N = 3 \) packets to broadcast to \( M = 3 \) receivers. Each receiver has RSI of the packets intended to the other \( M - 1 = 2 \) receivers. Then through network coding [1], where the 3 packets are combined to a single packet which is then broadcasted, all receivers are able to extract their desired packet in one transmission. In contrast to round robin scheduling (RRS) [8] where each packet is sent separately from T, this simple network coding operation brings about a 3-fold increase in the overall throughput.

Besides the ideal case shown in Fig. 1(a), the scenario where each receiver has different amount of RSI was considered in [2]. To maximize the throughput within one transmission, an opportunistic scheduling was proposed where only the packets destined for a subset of the receivers are combined and transmitted according to the RSI they have and the instantaneous channel conditions. That is, as shown in Fig. 1(b), only packets 1 and 3 are combined and transmitted from T such that receivers \( R_2 \) and \( R_3 \) can decode both packets 1 and 3 in one transmission. However, most of the times the receivers with less RSI (e.g. receiver \( R_1 \)) or worse channel conditions will not be served.

To the best of our knowledge, there is no published work discussing about how different amount of RSI available at the receivers will affect the overall network throughput of a broadcast channel. As shown in Fig. 1(b), if some receivers have RSI of only one packet or even no RSI, is it still beneficial to perform network coding? Specifically, what is the throughput advantage of network coding over RRS? In this paper, we analyze how different amount of RSI available at the receivers will affect the network throughput of a broadcast channel when network coding is performed.

In general, the accumulation of RSI from correctly decoded...
packets at each receiver is a random process. Thus instead of arbitrarily assuming certain packets as available RSI in a broadcast channel [2]-[7], we model the accumulation of RSI at each receiver \( R_m \) by a Poisson process with innovative packet arrival rate \( \lambda_m \). With different arrival rates \( \lambda_m \), we can thus model different amount of RSI available at each receiver.

We consider two transmission schemes for a broadcast channel, namely random linear network coding (RLNC) [9]-[13], and RRS [8]. To ensure reliable packet delivery, we consider an Automatic Repeat-reQuest (ARQ) scheme where the transmitter performs retransmissions until an acknowledgement (ACK) is sent back from every receiver indicating successful decoding of the desired packets. Taking into account the amount of RSI available at each receiver, closed-form expressions of network throughput for both RLNC and RRS are obtained by applying the renewal-reward theorem [14], [15]. The asymptotic throughput at high SNR is also characterized.

Our results show that the asymptotic throughput of RLNC is always no worse than RRS no matter how much RSI is available at each receiver. With a sum constraint of arrival rate \( \lambda_m \) over all receivers in the broadcast channel, the network throughput of RLNC is maximized if all \( \lambda_m \) are equal.

Notations: A circularly symmetric complex Gaussian random variable \( z \) with variance \( \sigma^2 \) is denoted as \( z \sim \mathcal{CN}(0, \sigma^2) \).

An exponentially distributed random variable \( x \) with mean \( \gamma \) is denoted as \( x \sim \exp(\gamma) \).

The mutual information of channel \( m \), \( t \) is given by

\[
I_{m,t} = \log_2 (1 + \rho \gamma_{m,t})
\]

where \( \rho = \frac{P_t}{\sigma^2} \) denotes the transmit SNR.

Due to the independent fading channels between the transmitter and multiple receivers, some receivers may not successfully decode the transmitted packet after a single transmission. To ensure reliable packet delivery, we consider a simple ARQ scheme where the transmitter performs retransmissions until an ACK is sent back from every receiver indicating successful decoding of the transmitted packets.

With the assistance of the available RSI, denoted by \( S_m \subseteq \mathcal{W}_T \), each receiver \( R_m \) attempts to decode all \( N \) packets in \( \mathcal{W}_T \). The arrival of \( |S_m| = k, k \in \{0, 1, \cdots, N\} \) innovative packets as RSI is modeled as a truncated Poisson process with probability mass function (pmf)

\[
f_i(k; \lambda_m) = \frac{\lambda_m^k e^{-\lambda_m}}{k!} \frac{1}{\sum_{k=0}^{N} \frac{\lambda_m^k e^{-\lambda_m}}{k!}} = \lambda_m^k \frac{1}{k!} \sum_{k=0}^{N} \frac{\lambda_m^k}{k!},
\]

and cumulative distribution function (cdf)

\[
F_i(k; \lambda_m) = \frac{k}{\sum_{i=0}^{N} \frac{\lambda_m^i}{i!}}.
\]

With different arrival rates \( \lambda_m \) of innovative packets, receiver \( R_m \) is able to accumulate different amount of RSI. The sum over all arrival rates \( \sum_m \lambda_m \) characterizes the total amount of RSI available in the broadcast channel on average. By modeling the accumulation of RSI at each receiver as an independent truncated Poisson process, we next investigate how different amount of RSI available at the receivers of a broadcast channel will affect the overall network throughput.

III. NETWORK THROUGHPUT ANALYSIS

We first define the event \( A_{m,t} = \{ \text{receiver } R_m \text{ has decoded all } N \text{ packets in } \mathcal{W}_T \text{ within } t \text{ retransmissions} \} \). Thus \( A_t = \bigcap_{m \in \{1, 2, \cdots, M\}} A_{m,t} \) corresponds to the event that all \( M \) receivers finish decoding all \( N \) packets within \( t \) retransmissions.

Since the receivers experience independent channels, event \( A_t \) occurs with probability

\[
\Pr(A_t) = \prod_{m \in \{1, 2, \cdots, M\}} \Pr(A_{m,t}).
\]

Then the probability that the broadcasting finishes at exactly the \( T_d \)th retransmission (and not earlier) is given by [6], [7]

\[
P_e(T_d) = \Pr(A_{T_d}) - \Pr(A_{T_d-1}).
\]

Fig. 2. A wireless broadcast channel where \( N \) packets are transmitted to \( M \) receivers which may have some receiver side information (RSI).
In order to obtain the network throughput which quantifies the average rate of successful packet delivery over a network, we assume that the transmitter T will initiate the transmission of a new set of N packets right after all M receivers have successfully decoded the previous set of N packets, and this transmission cycle will repeat itself indefinitely. For each transmission cycle, the number of retransmissions required for all M receivers to finish decoding, i.e., the download time, is given by $T_d$. The reward $\mathcal{R}$ of the broadcast channel for a single transmission cycle can be quantified by the sum rate of download time probability for link $T$. Essentially, the DoF of degrees of freedom (DoF) to decode all transmitted packets.

We assume that the reward $\mathcal{R}$ is equal for every transmission cycle and each channel is independent and identically distributed (i.i.d.) for every transmission. Thus we can apply the renewal-reward theorem [14], [15] to obtain the normalized network throughput as

$$\eta = \frac{\mathcal{R}}{E[T_d]} \quad \text{[bits/s/Hz]} \quad (5)$$

where

$$E[T_d] = \sum_{T_d=1}^{\infty} T_d P_c(T_d). \quad (6)$$

**Remark 1:** As long as the reward $\mathcal{R}$ is finite, through RLNC or RRS, receiver $R_m \forall m$ is able to finish decoding all $N$ packets within a finite number of retransmissions, thus the download time $T_d$ is finite and (6) converges.

**A. Network Throughput for RLNC**

For RLNC [9]-[13], we adopt an algebraic approach [9] and consider whether each receiver has collected sufficient degrees of freedom (DoF) to decode all transmitted packets. Essentially, the DoF of $\mathcal{W}_T$ is equal to the total number of innovative packets $N$. We may equivalently associate a vector space $\Omega$ to $\mathcal{W}_T$ where $\text{rank}(\Omega) = \text{dim}(\mathcal{W}_T) = N$. The available RSI $S_m \subseteq \mathcal{W}_T$ can be viewed as a vector subspace $\Omega_m \subseteq \Omega$ with DoF $\text{rank}(\Omega_m)$. Thus in order to decode all $N$ packets, $R_m$ with $|S_m| = k$ has to collect an additional $N - k$ DoF.

We define $A_{m,T_d}^{\text{rlnc}}$ as the event that $R_m$ has collected $N - k$ DoF to decode all $N$ packets within $T_d$ retransmissions. Denoting $s_{m,T_d}$ as the number of correctly received packets out of $T_d$ retransmissions, as a best-case lower bound $^4$, we have $A_{m,T_d}^{\text{rlnc}} = \{s_{m,T_d} \geq N - k\}$.

For simplicity, we assume that $R_i = R_0 \forall i$, then the outage probability for link $T \rightarrow R_m$ is given by

$$p_{m,t} = \Pr\{I_{m,t} < R_0\} \doteq p_m \quad (7)$$

where $p_{m,t}$ is independent of $t$. Thus the random variable $s_{m,T_d}$ follows a binomial distribution $B(T_d, (1 - p_m))$ with $T_d$ trials and success probability of $(1 - p_m)$. Hence, event $A_{m,T_d}^{\text{rlnc}}$ occurs with probability

$$\Pr\{A_{m,T_d}^{\text{rlnc}}\} = \sum_{k=0}^{N} \left[ \sum_{t=N-k}^{T_d} {T_d \choose t} (1 - p_m)^t p_m^{(T_d-t)} f_t(k; \lambda_m) \right]. \quad (8)$$

Then event $A_{m,T_d}^{\text{rlnc}} = \cap_{m \in \{1,2,\ldots,M\}} A_{m,T_d}^{\text{rlnc}}$ occurs with probability

$$\Pr\{A_{m,T_d}^{\text{rlnc}}\} = \prod_{m \in \{1,2,\ldots,M\}} \Pr\{A_{m,T_d}^{\text{rlnc}}\}. \quad (9)$$

Substituting (9) into (4)-(6), we can thus derive the network throughput for RLNC as

$$\eta^{\text{rlnc}} = \frac{\mathcal{R}}{\sum_{T_d=1}^{\infty} T_d \left[ \Pr\{A_{m,T_d}^{\text{rlnc}}\} - \Pr\{A_{m,T_d-1}^{\text{rlnc}}\} \right]}. \quad (10)$$

**B. Network Throughput for RRS**

For comparison purposes, we derive similar network throughput expressions for RRS. In RRS, retransmissions are carried out for a packet $W_i$, $i \in \{1,2,\ldots,N\}$, until it has been decoded by all $M$ receivers. After which, packet $W_{i+1}$ will be broadcasted by T. We assume that the innovative packets known a priori at a receiver are uniformly distributed, i.e., if receiver $R_m$ has $|S_m| = k$ innovative packets as RSI, then the probability that one of these packets is $W_i$ is given by $\Pr\{R_m \text{ has } W_i \mid |S_m| = k\} = \frac{k}{N}$. Thus from (1), the probability that $R_m$ has $W_i$ as RSI is given by

$$\Pr\{R_m \text{ has } W_i\} = \sum_{k=0}^{N} \Pr\{R_m \text{ has } W_i \mid |S_m| = k\} \cdot \Pr\{|S_m| = k\} = \frac{1}{N} \sum_{k=0}^{N} k \cdot f_t(k; \lambda_m) = \frac{E[|S_m|]}{N}, \quad (11)$$

where $E[|S_m|] = \sum_{k=0}^{N} k \cdot f_t(k; \lambda_m)$ denotes the average number of innovative packets known a priori at receiver $R_m$.

During the broadcasting of packet $W_i$, for those receivers that already have $W_i$, they will feedback to T immediately after the first transmission. For those receivers that do not have $W_i$, they will attempt to decode $W_i$ through maximal ratio combining (MRC) of all packets received over the retransmissions. T will continue transmitting until all M receivers have indicated that they have successfully decoded $W_i$.

We define $A_{m,T_i}^{\text{rrs}}$ as the event that $R_m$ finishes decoding $W_i$ within $T_i$ retransmissions. The probability that $R_m$ finishes decoding $W_i$ within $T_i = 1$ transmission is given by

$$\Pr\{A_{m,T_i=1}^{\text{rrs}}\} = \Pr\{R_m \text{ has } W_i\} + \Pr\{R_m \text{ does not have } W_i\} \Pr\{\log_2(1 + \rho \gamma_{m,1}) > R_i\}$$

$$= \frac{E[|S_m|]}{N} \left[ 1 - \frac{E[|S_m|]}{N} \right] \Pr\{\gamma_{m,1} > \frac{2R_i - 1}{\rho}\}. \quad (12)$$

Then the probability that $R_m$ finishes decoding for $W_i$ within $T_i \geq 2$ retransmissions is

$$\Pr\{A_{m,T_i}^{\text{rrs}}\} = \Pr\{R_m \text{ has } W_i\} + \Pr\{R_m \text{ does not have } W_i\} \Pr\left\{ \log_2 \left( 1 + \rho \sum_{t=1}^{T_i} \gamma_{m,t} \right) > R_i \right\}$$

$$= \frac{E[|S_m|]}{N} \left[ 1 - \frac{E[|S_m|]}{N} \right] \Pr\left\{ \sum_{t=1}^{T_i} \gamma_{m,t} > \frac{2R_i - 1}{\rho} \right\}. \quad (13)$$
Since $\gamma_{m,t} \sim \exp(\bar{\gamma}_m)$, we have from [16] \[ \Pr\{\sum_{t=1}^{T_i} \gamma_{m,t} > \frac{2^{t_i - 1}}{\rho}\} = 1 - F(G_i; T_i, \bar{\gamma}_m), \] where $F(\cdot)$ is the cdf of a Gamma distribution and $G_i = \frac{2^{t_i - 1}}{\rho}$. Then we can derive the network throughput for RRS as

\[ \text{Thus we can derive the network throughput for RRS as} \]

\[ \Pr\{A_{m,T_i}^{rrs}\} = \frac{E[|S_m|]}{N} + \left(1 - \frac{E[|S_m|]}{N}\right) \left[1 - F(G_i; T_i, \bar{\gamma}_m)\right]. \] (12)

Next, we define $A_{rrs} = \bigcap_{m \in \{1,2,\ldots,M\}} A_{m,T_i}^{rrs}$ as the event that all $M$ receivers finish decoding for $W_i$ within $T_i$ retransmissions, which occurs with probability

\[ \Pr\{A_{rrs}^{rrs}\} = \prod_{m \in \{1,2,\ldots,M\}} \Pr\{A_{m,T_i}^{rrs}\}. \] (13)

Substituting (13) into (4)-(6), we obtain

\[ E[T_{i}] = \sum_{T_i=1}^{\infty} T_i \left[\Pr\{A_{T_i}^{rrs}\} - \Pr\{A_{T_i-1}^{rrs}\}\right]. \]

Then the corresponding average download time required for all $M$ receivers to decode all $N$ packets in $W_i$ is given by

\[ E[T_d] = \sum_{i=1}^{N} E[T_i]. \] (14)

Thus we can derive the network throughput for RRS as

\[ \eta^{rrs} = \frac{N_{\text{RRS}}}{\sum_{i=1}^{N_{\text{RRS}}} T_i \left[\Pr\{A_{T_i}^{rrs}\} - \Pr\{A_{T_i-1}^{rrs}\}\right]}. \] (15)

\[ C. \text{ Asymptotic Throughput in High SNR Region} \]

1) RLNC: We denote $T_m$ as the download time for receiver $R_m$. From (7), $p_m \to 0$ when $\rho \to \infty$. Thus it requires only $\lim_{\rho \to \infty} T_m = N - |S_m|$ retransmissions for $R_m$ to decode all $N$ packets. Since the overall download time $T_d$ is limited by the worst receiver that finishes decoding last, we have

\[ \lim_{\rho \to \infty} T_d = \max_{m} \left(\lim_{\rho \to \infty} T_m\right) = N - \min_{m} (|S_m|). \] (16)

We let $Z = \min_{m} (|S_m|), Z \in \{0,1,\ldots,N\}$. At each receiver $R_m$, since the arrival of $|S_m|$ innovative packets is modeled as an independent truncated Poisson process given in (1) and (2), we have the cdf of $Z$ as

\[ F_Z(z) = 1 - \Pr \{Z > z\} \]

\[ = 1 - \Pr \{|S_1| > z\} \cdot \Pr \{|S_2| > z\} \cdots \Pr \{|S_M| > z\} \]

\[ = 1 - [1 - F_1(z, \lambda_1)] \cdot [1 - F_2(z, \lambda_2)] \cdots [1 - F_M(z, \lambda_M)]. \]

Then the corresponding pmf of $Z$ can be obtained by $f_Z(z) = F_Z(z) - F_Z(z - 1)$. Thus we have the average download time when $\rho \to \infty$

\[ E[\lim_{\rho \to \infty} T_d] = \sum_{z} (N - z) f_Z(z) = N - \sum_{z} z \cdot f_Z(z), \] (18)

and the corresponding asymptotic throughput is

\[ \lim_{\rho \to \infty} \eta^{\text{LinC}} = \frac{N}{N - \sum_{z} z \cdot f_Z(z)}. \] (19)

### Table I

**VARYING ARRIVAL RATES OF RSI WITH THE CONSTRAINT $\sum_m \lambda_m = \Lambda$**

<table>
<thead>
<tr>
<th>Arrival Rate</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\lambda_3$</th>
<th>$\lambda_4$</th>
<th>$\lambda_5$</th>
<th>$\min_m (\lambda_m)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario 1</td>
<td>0.2A</td>
<td>0.2A</td>
<td>0.2A</td>
<td>0.2A</td>
<td>0.2A</td>
<td>0.2A</td>
</tr>
<tr>
<td>Scenario 2</td>
<td>0.6A</td>
<td>0.1A</td>
<td>0.1A</td>
<td>0.1A</td>
<td>0.1A</td>
<td>0.1A</td>
</tr>
<tr>
<td>Scenario 3</td>
<td>$\Lambda$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

2) RRS: When $\rho \to \infty$, we have from (12)

\[ \lim_{\rho \to \infty} \Pr\{A_{m,T_{i=1}}^{rrs}\} = 1, \forall m, i, \text{ thus} \]

\[ \lim_{\rho \to \infty} \Pr\{A_{m,T_{i=1}}^{rrs}\} = \prod_{m \in \{1,2,\ldots,M\}} \lim_{\rho \to \infty} \Pr\{A_{m,T_{i=1}}^{rrs}\} = 1, \forall i \]

which means that it requires only $\lim_{\rho \to \infty} T_i = 1$ transmission for all $M$ receivers to successfully decode $W_i$. Then we can obtain the corresponding average download time

\[ E[\lim_{\rho \to \infty} T_d] = E[\lim_{\rho \to \infty} \sum_{i=1}^{N} T_i] = E[\lim_{\rho \to \infty} \sum_{i=1}^{N} T_i] = N, \] (21)

and the asymptotic throughput

\[ \lim_{\rho \to \infty} \eta^{rrs} = \frac{N}{N}. \] (22)

### IV. Simulation Results

For the broadcast channel shown in Fig. 2, we assume that the transmitter $T$ has a set of $N = 10$ innovative packets where $R_i = R_0 = 1 \forall i$ to broadcast to a group of $M = 5$ receivers. We let $\bar{\gamma}_m = 1 \forall m$. Thus for RLNC, the outage probability is given by

\[ p_{m,t} = 1 - e^{-2^{t_{m-1}} - \frac{1}{\rho}}, \forall m, t. \] (23)

To evaluate how different amount of RSI available at the receivers will affect the network throughput of a broadcast channel, we consider a sum constraint where $\sum_m \lambda_m = \Lambda$. In Table I, Scenario 1 represents the case where the arrival rate $\lambda_m$ is the same for all receivers. Scenario 2 represents the case where the arrival rate is skewed towards a single receiver ($R_1$). Scenario 3 is the extreme case where all RSI is concentrated in just one receiver ($R_1$).

In Fig. 3 and Fig. 4, the network throughput is shown for both RLNC and RRS under different scenarios with the constraint of $\Lambda = 20$ and 40 respectively. We can see that the theoretical results agree well with the simulation results. The network throughput of RLNC is bounded tightly by the corresponding asymptotic throughput given in (19). And for RRS, its network throughput is always upper bounded by

\[ \lim_{\rho \to \infty} \eta^{rrs} = \frac{N}{N} = 1 \] as given in (22).

An important observation from Fig. 3 and Fig. 4 is that the case where each receiver is able to accumulate the same amount of RSI (Scenario 1) achieves the maximum throughput for RLNC. Similar observation holds for RRS, though the difference in throughput is less significant. When some receiver is able to accumulate more RSI and others have little or even no RSI available (Scenarios 2 and 3), the network throughput of RLNC could be severely reduced since the download time...
is limited by the worst receiver with the least amount of RSI. However, we can see that the asymptotic throughput of RLNC is always no worse than RRS. This is because in RLNC, all packets are combined to a single packet which is then broadcasted. Thus the RSI available at each receiver can always be utilized to effectively reduce the number of transmissions required for successful decoding, as shown in (18). In contrast, for RRS where the packets are broadcasted separately, it requires at least $N$ transmissions for all receivers to decode all $N$ packets no matter how much RSI is available at each receiver, as shown in (21). Similar results can be observed in Fig. 4 where an overall higher network throughput is achieved for RLNC since the total amount of RSI available in the network, which is characterized by $\sum M \lambda_m = \Lambda$, is larger.

V. CONCLUSION

By modeling the accumulation of RSI at each receiver of a broadcast channel as an independent Poisson process, we have investigated how different amount of RSI available at the receivers will affect the overall network throughput. Two transmission schemes, random linear network coding and round robin scheduling, were considered. The corresponding network throughput was derived in closed-form expressions and the asymptotic throughput at high SNR was also characterized. Our results show that the asymptotic throughput of RLNC is always no worse than RRS no matter how much RSI is available at each receiver. With a sum constraint for the overall arrival rate of innovative packets in the broadcast channel, the case where each receiver is able to accumulate the same amount of RSI achieves the maximum throughput for RLNC.

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