Two-Way Relaying over OFDM: Optimized Tone Permutation and Power Allocation

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Abstract—We consider an amplify-and-forward scheme for two-way relaying over OFDM, in which two nodes wish to exchange information via a relay. Assuming full channel knowledge, we perform power allocation for the relay and both information-exchanging nodes, as well as tone permutation at the relay, so as to maximize the sum capacity. A dual decomposition technique is employed for power allocation, while a greedy approach is proposed for tone permutation. In particular, we explore an interesting water-filling behavior displayed by this power allocation solution. Numerical results demonstrate that substantial capacity gains are achieved by implementing the two proposed solutions, either individually or successively.

I. INTRODUCTION

Cooperative communications has recently received much renewed research interest, e.g., [1] among many others, although the study of relay channels dates back to the pioneering work carried out in the 1970s [2], [3]. In co-operative communications, the following scenario is commonly studied: a node, say $A$, wishes to send data to another node, say $B$, via a relay $R$. This one-way relaying can be represented as $A \rightarrow R \rightarrow B$.

If both nodes $A$ and $B$ wish to exchange information with each other via $R$, two-way relaying can instead be used [4], [5], known also as bi-directional relaying [6]. It is common to use two distinct phases for the protocol in two-way relaying [4], [5], but in general more distinct phases can be used [6]. We focus on a protocol with two distinct phases, comprising of a multiple-access (MA) phase and a broadcast (BC) phase. In the MA phase, both nodes concurrently send their information symbols to the relay, i.e., $A \rightarrow R \rightarrow B$. In the BC phase, the relay broadcasts the signals after some processing, i.e., $A \leftarrow R \rightarrow B$. Both phases are carried out over orthogonal radio resources, either over different time intervals in a time division duplex (TDD) mode, or over different frequency bands in a frequency division duplex (FDD) mode.

Similar to the one-way relay, the relay in the two-way relay can employ two basic schemes for processing the signal that has been received in the MA phase. In the decode-and-forward (DF) scheme, the relay tries to decode the information bits from both nodes. Then, the relay performs either a superposition or an XOR operation on the re-encoded bits and forwards the resulting output [4]–[6]. In [5], the use of multiple antennae has been further considered. Alternatively, in the amplify-and-forward (AF) scheme, the relay simply amplifies and forwards the received symbols [4]. In both DF and AF schemes, each node will cancel the self-interference originating from itself in the previous MA phase, before decoding the information bits from the other node.

As is common in the literature, we study the case that perfect self-interference cancelation is achieved, by assuming that channel state information (CSI) of all links is available at both nodes. If the direct channel between the two nodes is too weak (which likely motivates relaying in the first place), the CSI can be acquired in a handshaking stage by one-way relayings via the relay, e.g., $A \rightarrow R \rightarrow B$, then $A \leftarrow R \leftarrow B$. We further assume that the relay has CSI of all links. This knowledge can be obtained as a by-product of the handshaking stage, or by explicit feedback from both nodes.

In this paper, two-way relaying over $N$ parallel tones of an OFDM system is investigated. We consider that the relay uses only the AF scheme instead of DF scheme, applicable if the relay has no knowledge of the codebook used. However, we exploit the availability of the CSI of all links at the relay $R$ and nodes $A$, $B$. With this knowledge, our problem is to maximize the sum capacity by power allocation and tone permutation for two-way relaying. Specifically, $R$ allocates power across the tones subject to a sum-power constraint, and similarly for $A$ and $B$; further, $R$ permutes the frequency-domain data before transmission in the BC phase. This problem has been recently considered for one-way relay [7].

Our problem however becomes intractability for large $N$, thus we seek sub-optimal solutions suitable for practical implementations. For power allocation, we employ a dual-decomposition technique to solve the sum-capacity maximization problem, which is asymptotically optimal for large $N$ [8], [9]. This power allocation solution displays an interesting water-filling behavior. Specifically, at a given tone, if the product of the SNRs of both nodes is sufficiently small, then power is not allocated to them; if one SNR is small while the other is large, then power is allocated to only one node (i.e., one-way relaying is implemented). For tone permutation, we propose a greedy approach which performs close to the optimal permutation. Numerical results demonstrate that substantial capacity gains are achieved using the proposed solutions for power allocation and permutation, either individually or successively.

The paper is organized as follows. Section II describes the system model and states the power-allocation and tone-permutation optimization problem. Both problems are investigated in Section III and Section IV, respectively. Section V gives numerical results, while Section VI concludes the paper.
II. SYSTEM MODEL

A. Two-way Relay

The two-way relay, consisting of two nodes A and B exchanging information with each other via a relay R, is shown in Fig. 1. All transmissions are carried out over N parallel channels. In this paper, we consider an OFDM system in which these channels are subcarriers or tones. Our approach can be easily extended to a multiple-input-multiple-output (MIMO) OFDM system, by diagonalizing the channel with the use of a suitable transmit precoder, see e.g., [7].

For tone k = 1, · · · , N, the power of the channel coefficients from A to R are denoted as  ̂κk, from R to B as  ̂βk, from R to A as  ̂αk and from R to B as  ̂βk. In a TDD mode,  ̂αk =  ̂κk and  ̂βk =  ̂βk (we assume that channel reciprocity applies). In a FDD mode, these channel powers are usually different.

We use a AF scheme, and thus both the MA and BC phases use equal amount of channel resources, i.e., equal time duration for the TDD mode or equal bandwidth for the FDD mode. We now describe the phases of two-way relay for one tone. For brevity, we drop the tone index k.

1) Multiple-Access Phase: In the MA phase, A transmits the symbol xA using a power of a. Concurrently, B transmits the symbol xB using a power of b. The received signal at R is given by

\[ y_R = \sqrt{a}\alpha e^{J\theta}x_A + \sqrt{b}\beta e^{J\theta}x_B + n_R, \]

where nR is complex AWGN of unit variance, while  \( \theta \) is the phases of the complex channels from A to R and from B to R, respectively. Without loss of generality, we let  \( \theta \) and  \( \theta \) (and also the channel phases in the BC phase) be zero. This is valid because perfect cancelation of self-interference and coherent detection will be carried out at the receivers of A and B after the BC phase.

2) Broadcast Phase: In the BC phase, the relay performs power normalization  \( \eta \) to achieve a unit power, then allocates power  \( r \) for transmission at the  \( \pi \)th tone. Thus, the data originally at tone k is permuted to tone  \( \pi \); we call this tone permutation. The signals received at A and B are given as

\[ y_A = \sqrt{r}\tilde{\alpha}_k x_A + \eta x_R + n_A, \]
\[ y_B = \sqrt{r}\tilde{\beta}_k x_B + \eta x_R + n_B, \]

respectively, where  \( \eta = (a\tilde{\alpha} + b\tilde{\beta} + 1)^{-1/2} \) and  \( n_A, n_B \) are complex AWGN of unit variance.

Due to power allocation, the channel powers  \( \tilde{\alpha}_k \) and  \( \tilde{\beta}_k \) in the MA phase are effectively replaced by  \( a\tilde{\alpha}_k \) and  \( b\tilde{\beta}_k \), respectively, as shown in Fig. 1(a). Similarly,  \( \tilde{\alpha}_k \) and  \( \tilde{\beta}_k \) in the BC phase are effectively replaced by  \( r_k\tilde{\alpha}_k \) and  \( r_k\tilde{\beta}_k \), respectively, as shown in Fig. 1(b). Further, Fig. 1(b) illustrates that tone permutation changes the tone indices from  \( k \) to  \( \pi_k \).

B. SNR and Capacity

For notational convenience, we write  \( \pi_k \) as l. We assume perfect self-interference cancelation, i.e., A can perfectly remove from  \( y_A \) the signal component of  \( x_A \), and similarly for

\[ \begin{align*}
\text{(a) Multiple-access phase: nodes A, B transmit concurrently to relay R.} \\
\begin{array}{cccc}
A & \cdots & A & \cdots \\
\vdots & \ddots & \vdots & \ddots \\
A & \cdots & A & \cdots \\
\end{array}
\end{align*}
\]

\[ \begin{align*}
\text{(b) Broadcast phase: relay R permutes and broadcasts to nodes A, B,} \\
\begin{array}{cccc}
A & \cdots & A & \cdots \\
\vdots & \ddots & \vdots & \ddots \\
A & \cdots & A & \cdots \\
\end{array}
\end{align*}
\]

Fig. 1. The two-way-relay protocol over N parallel subcarriers. Power allocation is performed by R, A, B via \{r_k\}, \{a_k\}, \{b_k\}, respectively. Tone permutation is performed via \( \pi_1, \cdots, \pi_N \).

B. After perfect self-interference cancelation, the SNRs of the data first transmitted at tone k in the MA phase is given by

\[ \begin{align*}
\gamma_A,k(l, r_k, a, b) &= \frac{\tilde{b}_k\tilde{\beta}_k r_l\tilde{\alpha}_l}{\tilde{r}_l\tilde{\alpha}_l + z_{k}}, \\
\gamma_B,k(l, r_k, a, b) &= \frac{\tilde{a}_k\tilde{\alpha}_k r_l\tilde{\beta}_l}{\tilde{r}_l\tilde{\beta}_l + z_{k}},
\end{align*} \tag{1} \]

for A and B, respectively. Here, we denote

\[ z_k(a, b) = a\tilde{\alpha}_k + b\tilde{\beta}_k + 1. \tag{2} \]

For Gaussian codewords, the sum capacity for both nodes summed over all tones is given by

\[ C(\pi, r, a, b) = \sum_{k=1}^{N} c(k, l, r_k, a, b), \tag{3} \]

\[ c(k, l, r_k, a, b) \triangleq \left( \log(1 + \gamma_A,k) + \log(1 + \gamma_B,k) \right) / 2, \]

where the factor of half comes from the equal splitting of the MA and BC phases. Here,  \( \pi \) is the permutation vector with elements  \( \pi_k \) while  \( r, a, b \) is the power allocation vectors with elements  \( r_k, a_k, b_k \), respectively. In this paper, all logarithms have a base of two and all vectors are column vectors.

C. Problem Formulation

Our objective is to maximize the sum capacity  \( C \) by varying  \( \pi, r, a, b \). We subject the relay and each node to a sum-power constraint:  \( 1^T r \leq \bar{r}, 1^T a \leq \bar{a}, 1^T b \leq \bar{b} \), where  \( 1 \) is the all-one vector. A joint optimization of all variables concurrently quickly becomes computationally intractable as N increases. Instead, we consider two separate problems as follows.

P1: Fix  \( \pi \), optimize  \( C \) by varying  \( r, a, b \). See Section III.

P2: Fix  \( r, a, b \), optimize  \( C \) by varying  \( \pi \). See Section IV.

III. POWER ALLOCATION

Under P1, the permutation  \( \pi \) is fixed. Our optimization problem is to maximize the sum capacity (3) by varying the (non-negative) power allocation vectors  \( r, a \) and  \( b \), subject to
the sum-power constraints of \( r, \tilde{a}, \tilde{b} \), respectively, i.e.,
\[
\begin{align*}
\text{maximize} & \quad C(r, a, b) = \sum_k c(k, p_k) \\
\text{subject to} & \quad 1^T r \leq \bar{r},
\quad 1^T a \leq \bar{a}, \\
\quad 1^T b \leq \bar{b}.
\end{align*}
\]
(4a)

Here, \( p_k \triangleq [r_k/a_k, b_k]^T \) is the power-allocation vector for tone \( k \). Following [10], we call (4) the primal problem. We denote the optimal solutions as \( r^*, a^*, b^* \).

\[ \text{A. Dual Problem} \]

Let \( \lambda_r, \lambda_a, \lambda_b \) be the non-negative Lagrange multipliers that account for the power constraints of \( R, A, B \), respectively, in (4b). The Lagrangian associated to the primal problem is
\[
\begin{align*}
L(r, a, b, \lambda) = C(r, a, b) - \lambda_r (1^T r - \bar{r}) \\
- \lambda_a (1^T a - \bar{a}) - \lambda_b (1^T b - \bar{b}),
\end{align*}
\]
where \( \lambda \triangleq [\lambda_r, \lambda_a, \lambda_b]^T \). The corresponding dual function is
\[
f(\lambda) = \max_{r \geq 0, a \geq 0, b \geq 0} L(r, a, b, \lambda).
\]
(6)

Finally, the dual problem is defined as
\[
\text{minimize } f(\lambda).
\]
(7)

We denote the optimal solution as \( \lambda^d \). It is known that the duality gap \( d = f(\lambda^d) - C(r^*, a^*, b^*) \) is non-negative generally, but in particular if the primal problem is convex, then \( d = 0 \) [10]. However, our optimization problem (4) is non-convex, since the objective function (4a) can be shown to be non-convex, in which case the duality gap may be positive.

Even though the primal problem is non-convex, if a time-sharing condition is satisfied, the duality gap becomes asymptotically zero for large \( N \) [8]: a rigorous investigation for this phenomenon is given in [9]. Although an interference channel has been considered in [8], it can be shown that the problem formulation applies to our scenario of a two-way channel. Hence, it follows that the duality gap is zero asymptotically.

\[ \text{B. Dual Decomposition} \]

To simplify the dual problem, known here as the master problem, we employ dual decomposition. To this end, we write the dual function (6) as
\[
f(\lambda) = \sum_k f_k(\lambda) + \lambda_r \bar{r} + \lambda_a \bar{a} + \lambda_b \bar{b}
\]
(8)

where for \( k = 1, \ldots, N \), we have
\[
f_k(\lambda) = \max_{p_k \geq 0} c(k, p_k) - \lambda^T p_k.
\]
(9)

Thus, \( f(\lambda) \) has been decomposed as \( N \) maximization subproblems which can be independently solved given \( \lambda \).

\[ \text{C. Implementation Details} \]

The master problem (a convex optimization problem) is solved using a subgradient approach with an appropriate step size [10], which allows us to obtain \( \lambda^d \). In the process, we also obtain \( \{r, a, b\} \) which maximizes the Lagrangian \( L(r, a, b, \lambda^d) \) in (6). These power vectors are then normalized so that the power constraints (4b) is satisfied. This gives a feasible solution \( \{r^d, a^d, b^d\} \), where \( r^d = \bar{r} r / (1^T \bar{r}) \), \( a^d = \bar{a} a / (1^T \bar{a}) \), \( b^d = \bar{b} b / (1^T \bar{b}) \). Numerical results confirm that even for \( N = 32 \) (see Fig. 3), \( d_\lambda \equiv f(\lambda^d) - C(r^d, a^d, b^d) \approx 0 \). Since \( d_\lambda \geq d \geq 0 \), the duality gap \( d \) is also close to zero.

With dual decomposition, the subproblems can be solved independently. Solving each subproblem is however not trivial, as each subproblem is non-convex. Nevertheless, each subproblem requires the optimization of three variables and is still manageable by using an optimization software to perform a global search. This approach is adopted in Section V.

\[ \text{D. A Water-filling Phenomenon} \]

To develop intuition, we first give a numerical example. For simplicity, let us consider the TDD mode (i.e., \( \alpha = \beta = \beta \)) with \( N = 4 \) tones, at an average SNR of 0 dB. We use the set of (typical) channel realizations shown in Fig. 2(a). The solutions \( r^d, a^d, b^d \), obtained as described in Section III-C, are shown in Fig. 2(b). We observe that no power is allocated for tones 1, 3 in Fig. 2(b). From Fig. 2(a), we see that the channels at these tones have somewhat smaller SNRs, compared to other tones. This is reminiscent of the water-filling (WF) phenomenon typically observed in conventional power allocation problems: power is not allocated to a tone if its corresponding channel has SNR below a certain threshold. The distinction for the two-way channel is that for each tone, there is a pair of corresponding channels (in the FDD mode, the number of corresponding channels becomes four). We now give a characterization of the WF phenomenon for the two-way relay channel.

Consider subproblem (9) for tone \( k \). We take \( \lambda = [\lambda_r, \lambda_a, \lambda_b]^T \) as fixed, e.g., \( \lambda = \lambda^d \). For brevity, we drop index \( k \). Let \( \{\bar{r}, \bar{a}, \bar{b}\} \) be the optimal solution for the subproblem.

\[ \text{Fig. 2. Power allocation optimized using the dual function (right), based on typical SNRs (left) generated at an average SNR of 0 dB. Here, we use TDD mode and } N = 4 \text{ tones.} \]

\[ \text{\[a\] Typical SNRs.} \]

\[ \text{\[b\] Optimized power allocation} \]
(without power normalization). We want to determine the SNR pair \( \alpha, \beta \) for which the relay either transmits (i.e., \( \tilde{r} > 0 \)), or not transmits (i.e., \( \tilde{r} = 0 \)). To this end, we express \( \tilde{r} \) as (see Appendix)

\[
\tilde{r} = \begin{cases} 
  \tilde{r} > 0, & \text{if } \phi(0) > \delta; \\
  0, & \text{if } \phi(0) \leq \delta.
\end{cases}
\] (10)

Here, we denote the \( \delta = 2 \ln(2) \lambda_r \geq 0 \) (because \( \lambda_r \geq 0 \)) and the function \( \phi(r) \)

\[
\phi(r) = \left( \frac{1}{1 + \gamma_A} \frac{\partial \gamma_A}{\partial r} + \frac{1}{1 + \gamma_B} \frac{\partial \gamma_B}{\partial r} \right)_{a=\tilde{a}, b=\tilde{b}},
\] (11)

where \( \gamma_A, \gamma_B \) and the derivatives are evaluated at \( a = \tilde{a}, b = \tilde{b} \). Moreover, we define \( \tilde{r} \) as the solution of

\[
\phi(r) = \delta.
\] (12)

Using (11), \( \phi(0) \) required in (10) can be expressed as

\[
\phi(0) = \frac{\alpha \beta (a + b)}{a \alpha + b \beta + 1}.
\] (13)

We note that if \( \tilde{r} = 0 \), then \( \tilde{a} = \tilde{b} = 0 \), since if no relaying will be carried out at a certain tone, then there is no need for A or B to send any data at that tone. Conversely, if \( a = b = 0 \), then clearly \( \phi(0) = 0 \leq \delta \), and using (10) we get \( \tilde{r} = 0 \).

Generally, \( \tilde{a} \) and \( \tilde{b} \) are functions of both \( a \) and \( b \). It is thus difficult to explicitly determine if \( \phi(0) \) given by (13) will exceed \( \delta \) or not. Therefore, it is not immediately clear if the WF phenomenon exists and if so, how to characterize it quantitatively. Nevertheless, if the product of the SNRs \( \alpha \beta \) is sufficiently small, then \( \phi(0) \) must be smaller than \( \delta \). We claim this under the reasonable assumption that \( \tilde{a} \) and \( \tilde{b} \) are bounded by some positive constants \( \Phi_1, \Phi_2 \), respectively. To see this, note that

\[
\phi(0) \leq \frac{\alpha \beta (\Phi_1 + \Phi_2)}{a \alpha + b \beta + 1} \leq \alpha \beta (\Phi_1 + \Phi_2)
\]

where the first inequality is due to the assumption and the second inequality is due to \( \tilde{a} \geq 0, \tilde{b} \geq 0 \). Now if \( \alpha \beta \leq \delta (\Phi_1 + \Phi_2) \), then clearly \( \phi(0) \leq \delta \). Hence, if the product \( \alpha \beta \) is small enough, \( \tilde{r} = 0 \) according to (10). Thus, \( \tilde{a} = \tilde{b} = 0 \) and no transmission occurs for that tone. This points to one dominant underlying mechanism, namely the product of the pair of the SNRs \( \alpha \beta \), that leads to the WF phenomenon. From Fig. 2, we see that \( \alpha \beta \) is indeed very small for tones 1, 3.

Another interesting phenomenon that is observed in Fig. 2 is that for tone 4, power is allocated only for R and A, so equivalently we have a one-way relay. This occurs when either \( \alpha \) or \( \beta \) is small, but not both. Intuitively, this is because data can be efficiently transferred only in one direction. Analysis of this phenomenon is more involved.

IV. Tone Permutation

Under P2, we instead fix the power allocation but maximize the sum capacity by varying the permutation \( \pi \), i.e.,

\[
\max_{\pi} C(\pi) = \sum_k c(k, \pi_k).
\] (14)

The optimal permutation for the one-way relay is obtained by a simple ordering operation [7]. For the two-way relay, however, an exhaustive search of \( N! \) possible permutations appears to be needed, which incurs a complexity of \( O(N^N) \). To maintain a reasonable complexity for large \( N \), we propose a heuristic solution with a complexity of \( O(N^2) \) in this section.

A. Greedy-based Permutation

We proposed a greedy-based permutation, by successively searching for an appropriate pair of tones. In each iteration of this search, we first choose the tone in the MA phase that maximizes the sum SNR \( \alpha_k \tilde{\alpha}_k + b_k \tilde{\beta}_k \); equivalently, we maximize \( z_k \) defined in (2). Note that the power allocation has been taken into account in general. With the chosen MA-phase tone index \( k^* \) (say), we choose the corresponding tone \( l \) in the BC phase to maximize the capacity \( c(k^*, l) \). After this pairing is completed, both tones cannot be chosen again subsequently.

Algorithm 1 describes the permutation in detail. We assume that \( a_k \tilde{\alpha}_k, b_k \tilde{\beta}_k, r_k \alpha_k \) and \( r_k \beta_k \) are known a priori for all \( k \).

Algorithm 1 Greedy-based permutation.

- Initialization: the sets of available tones in the MA and BC phases are initialized respectively as

\[
S_{ma}(1) = \{1, \cdots, N\}; \ S_{bc}(1) = \{1, \cdots, N\}.
\]

- For \( i = 1, \cdots, N \):

  - Choose tone available in the MA phase as

    \[
    k_i^* = \arg \max_{k \in S_{ma}(i)} a_k \tilde{\alpha}_k + b_k \tilde{\beta}_k = \arg \max_{k \in S_{ma}(i)} z_k.
    \]

  - Choose tone available in BC phase as

    \[
    l_i^* = \arg \max_{l \in S_{bc}(i)} c(k_i^*, l).
    \]

  - Remove \( k_i^*, l_i^* \) from available sets:

    \[
    S_{ma}(i + 1) = S_{ma}(i) \setminus \{k_i^*\}; \ S_{bc}(i + 1) = S_{bc}(i) \setminus \{l_i^*\}.
    \]

- Finally, the permutation is described by \( (k_i^*, l_i^*) \) for all \( i \).

For a given iteration, Algorithm 1 first chooses \( k^* \), then chooses \( l^* \) given \( k^* \). With a higher complexity, we could have instead jointly chosen \( (k, l) \) to maximize \( c(k, l) \) in each iteration. Numerical results has however indicated that both methods give almost the same performance.

Algorithm 1 requires an overall complexity of \( O(N^2) \). To see this, note that we compute these operations in iteration \( i \):

- For all \( k \), compute \( z_k = a_k \tilde{\alpha}_k + b_k \tilde{\beta}_k + 1: \)
  - \( 2(N - i + 1) \) additions (include additions of ones).
- For all \( l \), compute the SNRs in (1) to obtain \( c(k_i^*, l) \):
  - \( 2(N - i + 1) \) multiplications and the same number of additions and divisions.
- Two maximizations over sets of \( N - i + 1 \) elements:
  - \( 2(N - i + 1) \) comparisons

Summing from \( i = 1 \) to \( i = N \) over all iterations, an overall complexity of \( O(N^2) \) is incurred, because \( \sum_{i=1}^{N-1} N - i + 1 = O(N^2) \). This shows that the order of complexity is quadratic.
We consider an OFDM system with $N = 32$ tones in the FDD mode. The frequency-domain channels are generated using 8 i.i.d Rayleigh distributed time-domain taps. The average SNR without power allocation (PA) is fixed to be the same for all receivers, i.e., $E[\alpha] = E[\alpha] = E[\beta] = E[\beta]$. The sum-power constraint is set as $\bar{r} = \bar{a} = \bar{b} = N$.

We divide the sum capacity by $N$ and use this per-tone capacity to evaluate performance. From Fig. 3, the best performance is given by a successively-optimized scheme: by solving for $P_2$, then for $P_1$. Specifically, we first (temporarily) fix the PA as uniform, i.e., $r = a = b = 1$, then determine the tone permutation $\pi$ by using the greedy-based permutation. With this $\pi$, we use the dual-decomposition technique to obtain a new PA. Compared to the benchmark in which an arbitrary $\pi$ and a uniform PA are used, we see from Fig. 3 that our scheme gives an improvement of about $3$ dB ($2$ dB) at a capacity of $0.5$ bit/symbol (2 bit/symbol, respectively).

In some scenarios, it may not be possible to concurrently implement both tone permutation and PA. For instance, regulations that impose a spectral mask on transmissions may forbid the use of PA, yet tone permutation can still be carried out. Using only either an optimized $\pi$ or an optimized PA also brings in about $0.5–2$ dB of improvement over the benchmark, as seen in Fig. 3. Hence, applying only one of the optimized solutions may also be useful in more restrictive scenarios.

In Fig. 3, we also plot the upper bounds when PA is optimized (given by $f(\lambda_k(r^d))/N$). We see that it is only marginally larger than the corresponding capacity (computed by $C(r^d, a^d, b^d)/N$), which implies that the duality gap is small. This numerically confirms that the dual-decomposition approach we adopt achieves a sum capacity that is within a small margin to the maximally-possible sum capacity.

VI. Conclusions

The problem of jointly maximizing the capacity by varying the power allocation and tone permutation becomes intractable for large number of tones. Thus, we considered the separate problem of optimizing the power allocation with the tone permutation fixed, and vice versa. Specifically, we employed the dual-decomposition technique to optimize the power allocation, for which we investigated a water-filling phenomenon; we also proposed a low-complexity greedy-based permutation. These approaches, when applied individually or successively, are shown to improve performance, compared to a uniform power allocation and an arbitrary tone permutation. Possible extensions include employing these approaches iteratively.

APPENDIX

**Proof of (10)**

We re-write subproblem (9) as (the index $k$ is dropped)

$$f(\lambda) = \max_{p=\{r,a,b\}^T \geq 0} c(p) - \lambda^T p = \max_{r \geq 0} g(r)$$

where we denote $g(r) = c(\hat{p}) - \lambda^T \hat{p}$ and $\hat{p} = [r, \hat{a}, \hat{b}]^T$. We want to obtain $\tilde{r} = \arg \max g(r)$ and show that it is given by (10). To this end, we first obtain the derivative of $g$ as

$$g'(r) \propto \phi(r) - \delta,$$

where $\delta = 2 \ln(2) \lambda_r$ and $\phi(r)$ is defined by (11). By writing out (15) in full, it can be easily verified that $\phi$ is a positive decreasing function, and so $g$ is a concave function.

Consider $\lambda_r > 0$ so that $\delta > 0$; the case of $\lambda_r = 0 (= \delta)$ implies that $g'(r)$ is a decreasing function, and thus $\tilde{r} = 0$ trivially. Now, consider $\phi(0) > \delta$ and $\phi(0) \leq \delta$ separately. If $\phi(0) > \delta$, then a solution $\tilde{r} > 0$ must exist from solving $\phi(\tilde{r}) = \delta$ (which is a decreasing function and so must intersect the positive constant $\delta$). Since $g$ is concave, $\tilde{r} = \arg \max g(r) = \tilde{r}$. If $\phi(0) \leq \delta$, then $\phi(r) \leq \phi(0) \leq \delta$ and so $g'(r) \leq 0$. Thus, $\tilde{r} = \arg \max g(r) = 0$. Combining both results, we get (10).

REFERENCES


