ARQ with Subcarrier Assignment for OFDM Systems

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Abstract—We consider an automatic-repeat-request (ARQ) scheme in OFDM systems. When a transmission fails, the transmitter repeats the data symbols in the retransmission, but on different subcarriers. The receiver then performs maximal-ratio combining (MRC) on the data symbols in the original and ARQ transmissions, before symbol detection is carried out. Our goal is to determine subcarrier assignment of the data symbols in the retransmission, such that utility functions which determine system-level performances are maximized. We show that when the utility function is Schur concave, the optimum solution is to assign the ARQ subcarrier with the largest SNR to the original subcarrier with the smallest SNR. Further, we propose heuristic solutions when some imposed constraints are relaxed.

I. INTRODUCTION

Orthogonal frequency division multiplexing (OFDM) is an effective solution for delivering high data rates over wireless channels that suffer from frequency-selective fading. A number of wireless standards such as IEEE 802.11a [1], WiMax [2] and long term 3G evolution [3] have adopted OFDM-based solutions for physical layer transmission. An OFDM system combats multipath fading with the use of a cyclic prefix that, in conjunction with the Fourier transform bases, converts the frequency selective fading channel into a set of parallel subcarriers experiencing flat fading. It is common to use an automatic repeat request (ARQ) mechanism in OFDM systems when a packet transmission fails. In this mechanism, the transmitter retransmits the data when it fails to receive an acknowledgment (ACK) or receives an explicit negative ACK.

Various ARQ schemes [4]–[7] for OFDM wireless systems have recently been proposed. In ARQ with subcarrier assignment, when an initial transmission fails, the frequency-domain OFDM symbols are retransmitted on different subcarriers. In [4]–[6], a diversity effect is realized by assigning data symbols to be retransmitted on ARQ subcarriers which are cyclic shifted versions of the original subcarriers. In [7], only ARQ subcarriers with SNR above a certain threshold are used for retransmission. However, so far the problem of choosing an optimal assignment was not considered. This is due in part to the difficulty in establishing a good utility function that accurately describes the system-level performance. Consequently, optimal schemes that fully exploit the channel state information (CSI) cannot be determined.

In this paper, we consider the problem of ARQ subcarrier assignment (ARQ-SA), of assigning data symbols that fail to be successfully received for retransmission in an OFDM system. Our goal is to optimize a utility function by appropriately choosing the assignment. We show that many utility functions of practical interest that we wish to maximize, such as capacity or probability of correct reception, are Schur concave. Schur concavity (or Schur convexity) has been recently investigated in the optimization of wireless systems [8]–[10] under the theory of majorization [11]; see also [12] for a review and applications of majorization theory.

We show that the optimum assignment for the ARQ-SA problem when the utility function is Schur-concave is as follows: assign the \( m \)-th-strongest-SNR subcarrier in the retransmission, to the \( m \)-th-weakest-SNR subcarrier in the original transmission, for all \( m \). That is, if the subcarriers in the original transmission are ordered with their respective SNRs in decreasing order, then the assigned subcarriers in the retransmission should be ordered with their respective SNRs in increasing order. This optimality is proven under two constraints, namely that each data symbol is retransmitted only in one subcarrier (hence the assignment becomes a permutation operation), and that full channel state information (CSI) is available for determining the assignment. If we relax the first constraint, the problem of determining the optimum assignment becomes an NP-hard problem. We hence propose a heuristic scheme. Further, to relax the second constraint, we use subcarrier grouping to reduce the amount of CSI required in implementation. Specifically, we group contiguous subcarriers and use an equivalent CSI for each group. The assignment is then carried out over each group of subcarriers, instead of over each subcarrier.

This paper is organized as follows. Section II describes the system model, while Section III describes the ARQ-SA problem. Section IV gives the mathematical preliminaries for Schur-concavity. In Section V, the optimal solution of the ARQ-SA problem under some constraints is given, while in Section VI heuristic solutions are proposed when the constraints are relaxed. Numerical results are given in Section VII to demonstrate the effectiveness of our proposed solutions. Finally, Section VIII concludes the paper.

II. SYSTEM MODEL

We consider an OFDM system as shown in Fig. 1. Due to the use of a cyclic prefix, data transmission can be considered to have occurred over \( M \) parallel subcarriers. As depicted...
in Fig. 1, data symbols $\{x_m\}$ are first transmitted in the original transmission. These data symbols are said to be in the frequency domain. The received signal can be expressed on a per-subcarrier basis as

$$r_m = h_m x_m + v_m$$  \hspace{1cm} (1)

for $m = 1, \ldots, M$, where $v_m$ is AWGN with unit variance. Here, $h_m$ is the frequency-domain channel coefficient in the original transmission, and we call it the $m$th original subcarrier.

Suppose that at least one bit in $\{x_m\}$ is not received correctly and ARQ is triggered. In the ARQ transmission, the symbol $x_m$ is assigned to be repeated on subcarriers with indices in the set $\{n \in A(m)\}$. In general, the number of assigned subcarriers in this set can be any number from zero to $M$, not necessarily only one. Similar to (1), the received signal in the ARQ transmission can be expressed on a per-subcarrier basis as

$$r'_n = g_n x_m + v'_n, \quad n \in A(m),$$  \hspace{1cm} (2)

for $m = 1, \ldots, M$, where $v'_n$ is also AWGN with unit variance. Here, $g_n$ is the frequency-domain channel coefficient in the ARQ transmission, and we call it the $n$th ARQ subcarrier.

At the receiver, we perform MRC for all the received signals that carry the same data symbol $x_m$ in (1) and (2) to get

$$\bar{r}_m = h^*_m r_m + \sum_{n \in A(m)} g^*_n r'_n,$$

for $m = 1, \ldots, M$. Subsequently, detection based on $\bar{r}_m$ is carried out to detect $x_m$. Consequently, the effective SNR of $x_m$ after MRC is given by summing the SNRs of the original subcarrier and the assigned ARQ subcarriers, given by

$$\gamma_m = \alpha_m + \sum_{n \in A(m)} \beta_n,$$  \hspace{1cm} (3)

for $m = 1, \ldots, M$. Here, we denote the power of the original subcarrier and ARQ subcarrier as $\alpha_m = |h_m|^2$ and $\beta_n = |g_n|^2$, respectively. Since we set all noise variances as one, $\alpha_m$ and $\beta_n$ are also the SNRs of the original and ARQ subcarrier, respectively.

For a time-invariant channel and when the original and ARQ transmission are carried out over the same frequency band, we have $h_m = g_m$. This scenario is commonly considered in the literature and is a special case of our model. Our model is more general which covers the case also when $h_m \neq g_m$, e.g., when the original transmission and ARQ transmission are carried out over different frequency bands.

### III. ARQ Subcarrier Assignment Problem

We denote the SNR vectors as $\alpha, \beta, \gamma$ with respective elements $\alpha_m, \beta_m, \gamma_m$, all of length $M$. The subcarrier assignment operation can be completely described by a binary $M \times M$ matrix $A$ with entries $a_{mn} \in \{0, 1\}$, as follows. If the signal in the $n$th original subcarrier is assigned to the $m$th ARQ subcarrier, we set $a_{mn} = 1$; otherwise, we set $a_{mn} = 0$. The effective SNR can then be written as

$$\gamma(A) = \alpha + A \beta.$$  \hspace{1cm} (4)

### A. Problem Statement

We want to find the optimal $A$ that maximizes a utility function $\phi(\gamma)$, with $\gamma$ given in (4). We call this the ARQ subcarrier assignment (ARQ-SA) problem.

### B. Utility Functions

The utility functions that suitably reflect system performance are:

$$\phi_{\min}(\gamma) = \min\{\gamma_1, \ldots, \gamma_M\},$$  \hspace{1cm} (5a)

$$\phi_{\max}(\gamma) = \sum_m \log(1 + \gamma_m).$$  \hspace{1cm} (5b)

When we use the utility function (5a), the ARQ-SA problem translates to maximizing the minimum effective SNRs among all subcarriers. The utility function based on mutual information in (6a), on the other hand, gives the amount of bits that can be reliably transmitted, assuming the use of Gaussian codebooks.

Another appropriate measure to minimize is the expected bit-error rate (BER), $P_e(\gamma_m)$, summed over all $m$. The utility function to maximize is then the negative of this measure. Assuming the use of QPSK modulation, the utility function is therefore

$$\phi_{\text{BER}}(\gamma) = -\sum_{m=1}^{M} Q\left(\sqrt{\gamma_m/2}\right),$$  \hspace{1cm} (5c)

The block error rate (BLER) is defined as the probability that at least one bit error occurs in a block. This is appropriate if a block is discarded when any bit error occurs. We can alternatively minimize the BLER. This is equivalent to maximizing the probability that all the bits in the block are successfully detected. Assuming the use of QPSK modulation, the utility function is therefore

$$\phi_{\text{BLER}}(\gamma) = \prod_{m=1}^{M} \left(1 - Q\left(\sqrt{\gamma_m/2}\right)\right).$$  \hspace{1cm} (5d)
IV. SCHUR-CONVEXITY

Before we further investigate the ARQ-SA problem, we show that the utility functions (5) satisfy a common property
given as follows.

Lemma 1: The utility functions (5) are Schur-convex.

Schur-convexity is closely related to the majorization theory
[11]; see the Appendix for the definitions of Schur-convexity
and majorization. To prove Lemma 1, we first state the following
lemmas that are useful for testing for Schur-convexity.

Lemma 2 ([11, Chap. 3.A.4]): Let \( A \subset \mathbb{R}^M \) be a symmetric,
non-empty convex set and let \( \phi : A \to \mathbb{R} \) be continuously
differentiable. The function \( \phi \) is Schur-convex on \( A \) if and
only if
\[
\phi \text{ is symmetric on } A, \quad \text{and} \quad (x_i - x_j) \left( \phi'_i(x) - \phi'_j(x) \right) \leq 0 \text{ for all } x \in A, \tag{6a}
\]

where \( \phi'_i(x) = \partial \phi(x) / \partial x_i \) denotes the partial derivative of \( \phi \)
with respect to \( x_i \).

Note that from Lemma 2, the following result immediately
follows (see also [11]).

Lemma 3: If \( \phi \) is symmetric and concave, then \( \phi \) is Schur-
convex.

We are now ready to prove Lemma 1.

Proof: We note that the utility functions (5) are symmetric,
i.e., interchanging the order of the arguments does not change
the output of the functions.

It is well known that \( \phi_M \) and \( \phi_m \) are concave functions.
Using Lemma 3 it then follows that these functions are Schur-
convex. Using standard calculus, the partial derivatives of the
remaining utility functions (5c), (5d) with respect to \( \gamma_m \) are
\[
\phi'_{\text{BER},m}(\gamma) = \frac{1}{2\sqrt{\pi}} \frac{\exp(-\gamma_m)}{\sqrt{\gamma_m}},
\]
\[
\phi'_{\text{BLER},m}(\gamma) = \frac{\exp(-\gamma_m/4)}{\sqrt{\gamma_m} \left( 1 - Q \left( \sqrt{\gamma_m} / 2 \right) \right)} h(\gamma),
\]

where \( h(\gamma) = \phi_{\text{BLER}}(\gamma) / (2 \sqrt{\pi}) \) is a symmetric function of
\( \gamma \). It can then be easily verified that (6b) is valid, hence \( \phi_{\text{BER}} \)
and \( \phi_{\text{BLER}} \) are Schur-convex functions by using Lemma 2.

V. SUBCARRIER ASSIGNMENT WITH CONSTRAINTS

In this section, we impose the following constraints for the
ARQ-SA problem.

C1: Each data symbol in the original transmission can only
be assigned to one subcarrier in the ARQ transmission.

C2: The transmitter has full CSI of the channels in the original
and ARQ transmissions.

We will consider heuristic solutions in Section VI when the
above constraints C1, C2 are relaxed.

Note that under constraint C1, the problem of determining
the optimum \( A \) reduces to finding the subcarrier permutation
(7) which maximizes \( \phi(\gamma) \).

A. Algorithm 1

We propose Algorithm 1 which performs a permutation \( \pi \)
such that the \( m \)th strongest ARQ subcarrier is assigned to the
\( m \)th weakest original subcarrier for all \( m \). In other words, if we
order the original subcarriers increasing (according to their
SNRs), then the correspondingly-assigned ARQ subcarriers
would be ordered decreasingly. Specifically, Algorithm 1 is
given as follows.

Algorithm 1

Initialization with inputs \( \alpha, \beta \):

- set \( A = 0 \);
- order \( \beta \) decreasingly to obtain \( \beta_1 \), so that \( \beta_n(1) \geq \cdots \geq \beta_n(M) \),
where \( n(l) \) is the ordered index;
- order \( \alpha \) increasing to obtain \( \alpha_1 \), so that \( \alpha_m(1) \leq \cdots \leq \alpha_m(M) \),
where \( m(l) \) is the ordered index.

Iteration \( l = 1, 2, \ldots, M \):
- assign ARQ subcarrier \( n(l) \) to original subcarrier \( m(l) \),
  i.e., set \( a_{m(l), n(l)} = 1 \).

By pairing strong ARQ subcarriers with weak original
subcarriers, Algorithm 1 produces effective SNRs which has
reduced fluctuations across subcarriers. Intuitively, we hence
expect that the minimum effective SNR \( \phi_{\text{min}} \), for instance, to be
increased compared to arbitrary pairings.

B. Optimality

The following result characterizes the optimum permutation
for all utility functions that are Schur concave.

Theorem 1: Let \( \alpha_1 \) be increasing ordered such that \( \alpha_1 \leq \cdots \leq \alpha_M \) and \( \beta_1 \) be decreasing ordered such that \( \beta_1 \geq \cdots \geq \beta_M \).

For a Schur concave function \( \phi \),
\[
\phi(\alpha_1 + \beta_1) \geq \phi(\alpha_1 + \beta_\pi)
\]

for any \( \beta_\pi \) which are permutations of \( \beta \).

Alternatively, we have \( \beta_1 = \arg \max_{\beta_\pi} \phi(\alpha_1 + \beta_\pi) \).

Proof: From [11, Chap. 6.A.2], we have \( \alpha_1 + \beta_1 \leq \alpha_1 + \beta_\pi \) for any \( \beta_\pi \) which are permutations of \( \beta \). It follows from
the definition of Schur convexity that (7) holds.

Theorem 1 shows that, for a Schur-convex function \( \phi \), the
optimal single ARQ-SA is to assign decreasingly-ordered
ARQ subcarriers to the increasingly-ordered original subcarriers.
We note that this is equivalent to Algorithm 1. Theorem 2
particularize this result to the utility functions (5) considered
in this paper.

Theorem 2: Algorithm 1 optimally solves ARQ-SA for the
utility functions (5) under constraints C1, C2.

Proof: Under C2, full CSI is available, while under C1, the
problem of ARQ-SA reduces to finding permutation \( \pi \) that
maximizes \( \phi(\gamma) \). By applying Lemma 1 and Theorem 1, the
result follows.

We note that the optimal solution for the ARQ-SA problem
may not be unique. This is because if two ARQ subcarriers
have exactly the same SNR, for instance, swapping their
assignments would not affect the effective SNRs or the utility functions.

VI. HEURISTIC SUBCARRIER ASSIGNMENT

A. Algorithm 2

We first relax constraint C1, but still impose constraint C2. Specifically, we allow each data symbol to be assigned to any number of ARQ subcarriers, e.g., zero, one, or several ARQ subcarriers.

We propose to obtain an assignment by using Algorithm 2, which imitates Algorithm 1, so as to maximize the utility function. We iteratively assign ARQ subcarriers to the original subcarriers, subcarrier by subcarrier in a greedy manner. Specifically, in the initialization stage, we set the effective SNR as the SNRs of the original subcarriers. Then, in each iteration, the strongest ARQ subcarrier that has not been assigned so far is assigned to the original subcarrier with the smallest effective SNR. After assignment, the effective SNR is updated in each iteration. Notice that this assignment may be carried out multiple times for the same original subcarrier, as long as its effective SNR remains smaller than the others.

Algorithm 2

Initialization with inputs $\alpha$, $\beta$:

- set $A = 0$ and $\gamma_0 = \alpha$;
- order $\beta$ decreasingly to obtain $\beta_1$, so that $\beta_{n(1)} \geq \cdots \geq \beta_{n(M)}$, where $n(l)$ is the ordered index.

Iteration $l = 1, 2, \ldots, M$:

- find smallest effective SNR in $\gamma_{l-1}$; denote its index as $m(l)$;
- assign ARQ subcarrier $n(l)$ to original subcarrier $m(l)$, i.e., set $a_{m(l),n(l)} = 1$;
- update the effective channel power as:

$$\gamma_l,m(l) = \gamma_{l-1,m(l)} + \beta_{n(l)}.$$

B. Optimality

We now establish that without constraint C1, the ARQ-SA problem is at least NP-hard for the utility function $\phi_{\min}$. To do so, we relate this problem with a known NP-hard problem. In [13], the following task allocation problem is considered: assign different tasks (or ARQ subcarriers, analogously) to processors (or original subcarriers), so that the minimum processor time (or SNR) required to complete all the tasks is maximized. This problem of [13] treats each processor as identical, while in the ARQ-SA problem the SNRs of the original subcarriers are not necessarily equal, and hence cannot be treated identically. Since the problem in [13] has been shown to be NP-hard, multiple ARQ-SA is at least NP-hard.

Algorithm 2 can be shown to have a implementation complexity of $O(M^2)$. Since this ARQ-SA problem (without constraint C1 but with constraint C2) is at least NP-hard, Algorithm 2 is not expected to be optimum. Nevertheless, Algorithm 2 is a simple solution that is suitable for implementation.

C. Grouping of Subcarriers

Next, we relax constraint C2 by limiting the CSI available, and consider heuristic assignments based on subcarrier grouping to solve the ARQ-SA problem. Specifically, we group $M$ contiguous subcarriers together and consider the ARQ-SA problem on these groups. If $G$ is much smaller than the coherence bandwidth of the channel, the subcarriers within each group have approximately the same SNR. Performing assignment based on the grouped subcarriers is thus likely to result in little performance loss. Within each group, a second (deeper) level of ARQ-SA can also be used; for instance, we can perform a fixed cyclic assignment [4]–[6] on the second level. Simulations conducted suggest that the additional performance gain that can be achieved is marginal and so a second level of ARQ-SA is not further considered.

For each group, we use a group-equivalent SNR to represent the SNR of the group. For our simulation studies, we use the minimum SNR of the group as the group-equivalent SNR, which is consistent with the minimum-SNR utility function (5a). Simulations reveal that using the minimum SNR as the group-equivalent SNR results in a better BER performance compared to using the arithmetic mean, and results in almost similar BER performance compared to using the geometric mean.

Although subcarrier grouping can be carried out with Algorithm 1 (with constraint C1) or with Algorithm 2 (without constraint C1), for simplicity we assume the former. Further, we assume that $M$ is divisible by $G$. The assignment with subcarrier grouping can then be described as Algorithm 3.

Algorithm 3

The same as Algorithm 1, but with $\gamma$, $\alpha$, $\beta$ replaced by their group-equivalent counterparts. Thus $A$ is a square matrix of reduced size $\tilde{M} = M/G$.

Clearly, Algorithm 3 reduces to Algorithm 1 when $G = 1$.

VII. NUMERICAL RESULTS

Simulations are conducted to illustrate the performances of using different assignments. We model the wireless channel with 16 i.i.d time domain channel taps. The channel is assumed to be time invariant during the original and ARQ transmissions. An error-free channel is used to signal the assignment used for ARQ-SA and the ACK/NACK for ARQ. We consider $M = 64$ subcarriers and employ zero-forcing equalization. We use the block-error rate (BLER) as performance measure.

As benchmarks, we consider two cases: firstly, no ARQ is employed, and secondly, cyclic assignment of the subcarriers in the ARQ transmission, as considered in [4]–[6]. In cyclic assignment, we use a cyclic shift of 16 to ensure that the SNRs of the original subcarrier and the corresponding ARQ subcarrier is highly uncorrelated. Lower bounds on BLER based on idealistic conditions are also plotted as dashed lines in Fig. 2. First, we consider the averaged matched filter bound (MFB). In averaged MFB, we assume that there is no
interference from other data symbols and that a matched filter is used for detection. Second, we employ the AWGN bound, where an AWGN channel is used, i.e., the original and ARQ subcarriers do not experience fading. To obtain both averaged MFB and AWGN bounds, ARQ is always activated.

From Fig. 2, we observe that Algorithms 1, 2, 3 are significantly better than the benchmarks (with no ARQ or cyclic assignment). Further, Algorithm 1 and 2 are within 0.5 dB away from the lower bound given by the averaged MFB. On the other hand, the difference in performance between Algorithm 1 and 2 is small. This is because wireless channels have high frequency selectivity, and since the weakest subcarriers are already significantly boosted by the strongest subcarriers using Algorithm 1, any further gain by using a more sophisticated algorithm, such as Algorithm 2, is likely to be small.

Finally, we observe that Algorithm 3 with $G = 2$ is at a few tenths of dB away from Algorithm 2, while Algorithm 3 with $G = 4$ is 1 to 1.5 dB away from Algorithm 2. This is because for $G = 2$, the subcarriers within the group have almost the same SNRs, but for $G = 4$ the subcarriers within the group can be quite different. For this particular wireless scenario, Algorithm 3 with $G = 2$ thus appears to offer a good tradeoff in terms of performance and amount of CSI required.

VIII. CONCLUSION

In this paper, we have considered several subcarrier assignment algorithms for ARQ in OFDM wireless systems. We have proven the optimality of Algorithm 1 under some constraints, and have considered heuristic algorithms when the constraints are relaxed. We have considered a rich class of utility functions that are Schur-concave. As such, we expect the algorithms we have proposed to work well for many other reasonable utility functions. As future work, we will develop and extend our results to other communication scenarios of interest and further relax the constraints.

APPENDIX

Definition of Majorization: For any $x, y \in \mathbb{R}^M$, we say that $x$ is majorized by $y$ (or $y$ majorizes $x$), denoted as $x \prec y$, if

$$\sum_{m=1}^{k} x_{[m]} \leq \sum_{m=1}^{k} y_{[m]}, \quad 1 \leq k \leq M - 1, \quad (8a)$$

$$\sum_{m=1}^{M} x_{m} = \sum_{m=1}^{M} y_{m}. \quad (8b)$$

Here, the subscript $[m]$ denotes a decreasing ordering such that $z[1] \geq z[2] \geq \cdots \geq z[M]$. Majorization thus defines a form of inequality among vectors.

Definition of Schur concave functions: A real-valued function $\phi$ defined on a set $A \subseteq \mathbb{R}^M$ is said to be Schur concave on $A$ if

$$x \prec y \text{ on } A \Rightarrow \phi(x) \geq \phi(y). \quad (9)$$

Thus, a Schur concave function $\phi$ defined on vectors from a set $A$ achieves its maximum at a vector that is majorized by all other vectors in the set.

REFERENCES