Dynamical properties of continuous attractor neural network with background tuning

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\textbf{Abstract}

Persistent activity holds the transient stimulus for up to many seconds even after the stimulus is gone. It has been implemented in a class of models known as continuous attractor neural networks, which have infinite stable states corresponding to persistent activity patterns. Continuous attractor neural network remains stable so does not change systematically in the absence of stimulus input. Continuous attractor is a set of connected stable equilibrium points and has been used to describe the storing of continuous stimuli in neural networks. The background input of the networks plays an important role in continuous attractor neural network. In this paper, dynamical properties of continuous attractor neural network with two background input tuning schemes are investigated: constant input shifting and oscillation background activity. Simulations are employed to illustrate the theory.

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1. Introduction

Persistent activity persisting for up to many seconds even after the transient sensory stimulus is gone is called persistent activity [1]. It is thought to be the neural substrate for short-term memory in a wide variety of brain areas [2]. Persistent activity is considered to be a neuronal correlate of working memory [3] and interpreted as the memory of eye position [4], head direction [5] and so on.

Persistent activity has been implemented in a class of models known as attractor neural networks, which have multiple stable states corresponding to persistent activity patterns. Attractor neural networks may function as memory devices. To do so, memories are encoded in the synaptic connections of the network as shown in [6–8], multiple patterns can be implemented as fixed-point attractors of the networks. An initial state may lead to dynamical flow into one of the attractors and thus recalls the stored pattern [9–12].

In graded persistent activity, neurons can sustain firing at many levels, suggesting a wide type of networks can relax to any one of a continuum of stationary states [13]. Such a continuum of stationary states is usually called continuous attractors.

Continuous attractor is a set of connected stable equilibrium points or continuous manifold of fixed-points. Continuous attractor neural network can be an integrator for any stimulus which causes the networks state to shift along the attractor. Once the stimulus is removed, the network remains stable, so does not change systematically in the absence of input [14]. Continuous attractors have been used to describe the encoding of continuous stimuli such as the eye position [4,15], head direction [5], the moving direction [16,17], path integrator [18–20], cognitive map [21] and population decoding [22,23]. Moreover, continuous attractor networks are able to maintain a localized packet of neuronal firing activity [24] and were used to store a pair of correlated maps such as a morph sequence between two uncorrelated maps [25]. Continuous attractors have some different shapes. Attractor that forms a loop is called a ‘ring attractor’; otherwise, attractor that does not loop back on itself is called ‘line attractor’ [26]. Ring attractor is used to represent periodic angular variables such as direction [27]. In order to study ring attractor, the neurons are aligned and the firing rates are bell-shaped or bump-shaped function of the stored variable, so ring attractor is also called bell-shaped attractor [28]. Line attractor is often applied to the memory of eye position [29,30].

The background input plays an important role in the performance of the neural networks. It may act as a switch that allows networks to be tuned on or off [31]. Small changes in the background input level may shift a network from a relatively quiet state to some other state with highly complex dynamics.
In [3], with an increased background input, the target population of neurons reactivates spontaneously with a set of population spike, and a further increase in background input leads to working memory with asynchronous elevated firing in the target population. In addition, periodic membrane oscillations due to the rhythmic background activity are typical for various brain regions, such oscillations may play a beneficial role in content-addressable memory processes [11]. Theta oscillations have been recorded in hippocampus [32]. Temporal correlations of active cells in medial septum and the hippocampal systems indicate that the medial septum provides a constant cholinergic modulation that facilitates oscillations and induces a phasic drive [33]. Successful memory formation is correlated with tight modulation that facilitates oscillations and induces a phasic drive [34].

We found some novel and interesting results in this paper. Two different tuning mechanisms are investigated in detail: constant background tuning and periodic background oscillations tuning. We found some novel and interesting results which are helpful for us to understand the essence of continuous attractor.

This paper is organized as follows. Linear neural network model is investigated in Section 2. Model with constant background activity is studied in Section 3. Model with oscillation background is studied in Section 4. Finally, conclusions are drawn in Section 5.

2. Neural networks model

Linear neural network is the simplest recurrent network model. In order to find out the dynamics properties of neural network with background tuning, we study the linear recurrent neural networks

\[ x(t) + x(t) = WX(t) + b(t) \]  

for \( t \geq 0 \), where \( x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \) is the state vector, \( x_i \) is the activity of neuron \( i \), \( W \) is the synaptic connection matrix which is obtained at the coding or training stage according to the Hebbian rule, we assume that \( W = (W_{ij})_{n \times n} \) is a symmetric real constant matrix. \( b(t) = (b_1(t), \ldots, b_n(t))^T \) denotes the background input which is independent on the initial conditions. Moreover, \( b(t) \) can be constant variable and function of time \( t \).

The linear network with constant background was investigated in our previous work [35] and the clear representation of continuous attractor was given. As an extension of the work, we focus on the continuous attractor shifting properties of linear network with this kind of constant background. Then, we study the dynamics properties of network when the background input is the periodic function of time \( t \) and find some other interesting results.

Since the synaptic connection matrix \( W \) is symmetric, it possesses an orthonormal eigensystem. Let \( \lambda_i (i = 1, \ldots, n) \) be the eigenvalues of \( W \) ordered by \( \lambda_1 \geq \lambda_2 \cdots \geq \lambda_n \). Suppose that \( S_i (i = 1, \ldots, n) \) compose an orthonormal basis in \( \mathbb{R}^n \) such that each \( S_i \) is an eigenvector of \( W \) belonging to \( \lambda_i \). Let the multiplicity of \( \lambda_1 \) be \( m \) and denote by \( V_{\lambda_1} \) the eigensubspace associated with the eigenvalue \( \lambda_1 \).

Given any initial state of the network \( x(0) \in \mathbb{R}^n \), let \( x(t) \) be the trajectory starting from \( x(0) \), then \( x(t) \) can be represented as

\[ x(t) = \sum_{i=1}^{n} z_i(t)S_i \]  

for \( t \geq 0 \), where \( z_i(t) (i = 1, \ldots, n) \) are some functions. In fact, they are the projections of the trajectory \( x(t) \) on the eigenvectors \( S_i \). It is clear that

\[ x(0) = \sum_{i=1}^{n} z_i(0)S_i. \]

Suppose that

\[ b(t) = \sum_{i=1}^{n} b_i(t)S_i \]

are the projections of the background input \( b(t) \) on the eigenvectors \( S_i \).

3. Models with constant background activity

In this section, the background input is constant and \( b(t) = b \), the elements of \( b \) can be positive, zero and negative. Suppose that

\[ b = \sum_{i=1}^{n} b_iS_i. \]

The theorem in our previous work [35] gives sufficient conditions for network (1) to possess continuous attractor.

**Lemma 1** (Yu et al. [35]). Suppose \( \lambda_1 = 1 \) and \( b \perp V_{\lambda_1} \). Then, the linear network (1) has a continuous attractor and the continuous attractor can be represented by

\[ C = \left\{ \sum_{i=1}^{n} c_iS_i + \sum_{j=m+1}^{n} \frac{b_j}{1-\lambda_j}S_i \mid c_i \in \mathbb{R} (1 \leq i \leq m) \right\}. \]

If all the eigenvalues of \( W \) is 1 with multiplicity \( n \), that is to say \( V_{\lambda_1} = \mathbb{R}^n \). Because \( b \perp V_{\lambda_1} \), then \( b = 0 \). In this case (1) can be rewritten as

\[ \dot{x}(t) = 0 \]

for \( t \geq 0 \).

So we get \( x(t) = x(0) \). The trajectory remains at the initial point. It means that all points in \( V_{\lambda_1} \) are equilibria.

On the other hand, if all the eigenvalue of \( W \) is less than 1, then the attractors of network are discrete but not continuous. The network with this kind of \( W \) does not belong to continuous attractor neural network.

So generally speaking, when the largest eigenvalue of \( W \) is 1 and the others are less than 1, the network possess continuous attractor. Moreover, the dimension of \( C \) is \( m < n \). \( C \) is a low-dimensional manifold embedded in the \( n-D \) state space. In the neural coding of eye position, the manifold representing the eye position in the brain is one dimension. The manifold example which has a high dimensionality more than one is the coding of the image. If we vary the orientation, translation and scaling of a face simultaneously, then we will get a three-dimensional facial images set.

We can derive that the continuous attractor \( C \) is in \( V_{\lambda_1} \), which is spanned by the eigenvectors of \( W \) with eigenvalues 1. We call the direction on the linear combination of the \( m \) eigenvectors with eigenvalues 1 the tangent direction of \( C \) and any direction orthogonal to the \( C \) is the normal direction. We should be noted...
that although each point in C is stable, it does not mean that all points in C are asymptotically stable.

**Definition 1.** An equilibrium point \( x^* \) is said to be asymptotically stable, if \( x^* \) is stable, and there exist a \( \eta > 0 \) such that 

\[ \| x(0) - x^* \| \leq \eta \]

implies that 

\[ \lim_{t \to +\infty} x(t) = x^* \]

for all \( t \geq 0 \).

**Theorem 1.** Suppose \( \lambda_1 = 1 \) is the largest eigenvalue of \( W \) with the multiplicity \( m \) and \( b \perp V_{\lambda_1} \), then the continuous attractor is only asymptotically stable on the normal directions. The trajectory starting from \( x(0) \) converges to 

\[ x^t = \sum_{i=1}^m z_i(0)S_i + \sum_{j=m+1}^n \frac{\tilde{b}_j}{1-\lambda_j} S_j. \]

**Proof.** Since \( b \perp V_{\lambda_1} \), then \( \tilde{b}_1 = \cdots = \tilde{b}_m = 0 \).

Since the multiplicity of the largest eigenvalue \( \lambda_1 = 1 \) is \( m \), it follows from (1) and (2) that

\[ z_i(t) = 0, \quad 1 \leq i \leq m \]

and

\[ z_j(t) = (\lambda_j-1) \cdot z_j(t) + \tilde{b}_j, \quad m+1 \leq j \leq n \]

for \( t \geq 0 \). It is easy to see that the derivative of \( z_i(t) \) is zero on the tangent direction of C. Solving this equation, it gives that

\[ z_i(t) = z_i(0), \quad 1 \leq i \leq m \]

and

\[ z_j(t) = \frac{\tilde{b}_j}{1-\lambda_j} + \left( z_j(0) - \frac{\tilde{b}_j}{1-\lambda_j} \right) e^{\lambda_j t}, \quad m+1 \leq j \leq n \]

for \( t \geq 0 \). Thus

\[ x(t) = \sum_{i=1}^m z_i(0)S_i + \sum_{j=m+1}^n \frac{\tilde{b}_j}{1-\lambda_j} S_j + \sum_{j=m+1}^n \left( z_j(0) - \frac{\tilde{b}_j}{1-\lambda_j} \right) S_j e^{\lambda_j t} \]

(3)

for \( t \geq 0 \). We can get from (3) that when \( t \to +\infty \), the third term on the right hand of (3)

\[ \sum_{j=m+1}^n \left( z_j(0) - \frac{\tilde{b}_j}{1-\lambda_j} \right) S_j e^{\lambda_j t} \to 0, \]

so

\[ x(t) \to \sum_{i=1}^m z_i(0)S_i + \sum_{j=m+1}^n \frac{\tilde{b}_j}{1-\lambda_j} S_j, \]

when \( t \to +\infty \). Then from **Definition 1**, the trajectories converge to the point on the continuous attractor C asymptotically only on the normal direction of C. The proof is complete. \( \square \)

Consider the model:

\[ \dot{x} + x = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} x + \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}. \]

(4)

Denote

\[ W = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, \quad b = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}. \]

It can be checked that the eigenvalues of \( W \) are \( \lambda_1 = 1 \) and \( \lambda_2 = 0 \). The eigenvectors of \( W \) belong to the above eigenvalues are

\[ S_1 = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}, \quad S_2 = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}. \]

Moreover, \( b \perp V_{\lambda_1} \).

The continuous attractor is

\[ C = \{ c \cdot S_1 + b : c \in \mathbb{R} \}. \]

The dashed line in Fig. 1 is the continuous attractor \( C \). \( S_1 \) is the tangent direction of the continuous attractor \( C \). If the initial point is \((1,1)\), the converge point is the projection of \((1,1)\) on \( C \) and the trajectory converges to this point asymptotically only on the normal direction of \( C \). If we give the initial point \((1,1)\) a change along the direction parallel to \( C \) to get another point \((0.5,0.5)\), the terminal point is also on the continuous attractor and is the projection of \((0.5,0.5)\) on \( C \). If we change the initial point along the normal direction which is orthogonal to \( C \) to another point \((1.5,0.5)\), then the converge point does not change. The reason is that the projections of the two initial points \((1,1)\) and \((1.5,0.5)\) on the continuous attractor are same. If we change the initial point along the direction which is neither tangent nor normal direction to \((2,1.5)\) continuously, then the steady state shifted continuously along the continuous attractor. The dotted line is the shift of initial points and the bold solid line is the shift of continuous attractor. The thin solid lines are the trajectories from different initial points.

Let us study the shift reduced by background shift. If the background \( b \) is changed, then the continuous attractor will translate but the direction will not change.

**Theorem 2.** Suppose \( \lambda_1 = 1 \) and \( b \perp V_{\lambda_1} \). Given \( b \) a small change continuously along the normal direction of the continuous attractor, then the continuous attractor \( C \) is shifted continuously to another continuous attractor along the normal direction of \( C \).

**Proof.** Given \( b \) a small change and \( b^* \) is obtained, where

\[ b = \sum_{i=1}^n \tilde{b}_i S_i. \]

Fig. 1. The shift of the initial points of (4) along different directions respectively.
and
\[ b^i = \sum_{i=m+1}^{n} b^i_j S_i. \]

Then the continuous attractor
\[ C = \left\{ \sum_{i=1}^{m} c_i S_i + \sum_{j=m+1}^{n} \frac{b_j}{1-l_j} S_j \mid c_i \in R, 1 \leq i \leq m \right\} \]
is shifted to
\[ C' = \left\{ \sum_{i=1}^{m} c_i S_i + \sum_{j=m+1}^{n} \frac{b_j}{1-l_j} S_j \mid c_i \in R, 1 \leq i \leq m \right\}. \]

We can see that the first \( m \) projections on the eigenvectors belong to eigenvalue 1 of \( C \) and \( C' \) are same, so \( C \) shift to \( C' \) continuously only along the normal direction of \( C \). The proof is complete. \( \square \)

Reconsider the model (4), when we increase the background input \( b \) along the normal direction of the continuous attractor and obtain other two background inputs: \( b_2 = 1.5b, b_3 = 2b \). Thus all the three inputs are orthogonal to \( V_{j,i} \). The simulation results are given in Fig. 2. The continuous attractor and trajectories of network with background input \( b \) are plotted in red. The continuous attractors and trajectories of network with background input \( b_2 \) and \( b_3 \) are plotted in blue and black, respectively. We can see that the continuous attractor is shifting continuously from the initial one to the final one and all the continuous attractors are parallel lines. The different projections on the \( S_i \) for \( m = 1, \ldots, n \) of the three inputs made this translation of continuous attractors.

4. Model with oscillating background input

Generally, when the background input is not constant but some continuous periodic vector function, the attractor is not a unique equilibrium but a periodic trajectory. In fact, periodic oscillations are prominent in the hippocampus. In this section, we investigate the dynamics properties of continuous attractor neural network with periodic background. Actually, the analysis of oscillation is more general than stability analysis since a fixed point is the special case of oscillation with any arbitrary period. We will show that under some conditions the network (1) has infinite stable periodic trajectories and has the same period as the background input.

The background input of the linear neural network is some continuous periodic vector function defined on \( [0, +\infty) \) with period \( \omega \), i.e., there exits a constant \( \omega > 0 \) such that \( b(t + \omega) = b(t) \) \( (i = 1, \ldots, m) \) for all \( t \geq 0 \).

Give any \( x \in \mathbb{R}^n \), we define a normal by
\[ |x| = \max_{1 \leq i \leq n} |x_i|. \]

Let \( D \subset \mathbb{R}^n \). For any initial point \( \phi \in D \), we denote the solution of (1) starting from \( \phi \) by \( x(t, \phi) = (x_1(t, \phi), x_2(t, \phi), \ldots, x_n(t, \phi))^T \). It means that \( x(t, \phi) \) is continuous, satisfies (1) and \( x(0, \phi) = \phi \).

Definition 2. A set \( D \) is called an invariant set, if each trajectory starting from \( D \) will stay in \( D \) forever.

Lemma 2 (Yu et al. [37]). Given any \( \phi, \psi \in D \), where \( D \) is an invariant set of the network (1), if constants \( \gamma > 0 \) and \( \epsilon > 0 \) exist such that
\[ |x(t, \phi) - x(t, \psi)| \leq \gamma |\phi - \psi|e^{-\epsilon t} \]
for all \( t \geq 0 \), then there is only one periodic trajectory of the network (1) in \( D \), which exponentially attracts all trajectories of \( D \).

Compared with the constant background, we found that the linear network with oscillating background has similar but more interesting results.

Theorem 3. Suppose \( \lambda_1 = 1 \) is the largest eigenvalue of \( W \) with the multiplicity \( m \) and \( b \perp V_{j,i} \), then the set \( D \perp V_{j,i} \) is an invariant set of the linear network (1). Moreover, the linear network (1) has only one periodic trajectory located in \( D \), it exponentially attracts all trajectories of \( D \). Thus, linear network (1) has infinite exponentially stable periodic trajectories.

Proof. The proof will be divided into two parts. In the first part, we will prove that \( D \) is an invariant set, i.e., given any initial \( x(0) \in D \), the trajectory \( x(t) \) starting from \( x(0) \) will stay in \( D \).

Since \( b(t) \perp V_{j,i} \), then \( \dot{b}_1(t) = \cdots = \dot{b}_m(t) = 0 \).

Since the multiplicity of the largest eigenvalue \( 1 \) is \( m \), it follows from (1) and (2) that
\[ z_i(t) = 0 \quad (i = 1, \ldots, m) \]
for \( t \geq 0 \). Then, we have
\[ z_i(t) = z_i(0) \quad (i = 1, \ldots, m) \]
for \( t \geq 0 \).

Since \( x(0) \in D \) and \( D \perp V_{j,i} \), then \( z_1(t) = \cdots = z_m(t) = 0 \).

So, \( z_i(t) = 0 \quad (i = 1, \ldots, m) \).

Next, the second part, we will prove that there is one periodic trajectory located in \( D \) and this periodic trajectory exponentially attracts all the trajectories of \( D \).

Let \( x(t, \phi) \) and \( x(t, \psi) \) be the two trajectories of the network (1) with initial conditions \( \phi \) and \( \psi \), respectively, where \( \phi, \psi \in D \). Then from (1), we have
\[ \ddot{x}(t, \phi) + Wx(t, \phi) + b(t) \]
and
\[ \ddot{x}(t, \psi) + Wx(t, \psi) + b(t) \]
for all \( t \geq 0 \). Moreover
\[ x(t, \phi) = \sum_{i=1}^{n} z_i(t, \phi)S_i \]

![Fig. 2. The shift of the continuous attractor of linear networks (4) with three different background inputs. The continuous attractors and trajectories of network with background input \( b \) are plotted in red, blue and black, respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)](image-url)
and
\[ x(t, \psi) = \sum_{i=1}^{n} z_i(t, \psi) S_i \]
for \( t \geq 0 \). Denote
\[ u(t) = x(t, \phi) - x(t, \psi) \]
and
\[ z_i(t) = z_i(t, \phi) - z_i(t, \psi) \quad (i = 1, 2, \ldots, n) \]
it follows that
\[ \dot{u}(t) + u(t) = W u(t). \]

Then we have
\[ \sum_{i=1}^{n} z_i(t) S_i + \sum_{i=1}^{n} z_i(t) S_i = W \sum_{i=1}^{n} z_i(t) S_i \]
for all \( t \geq 0 \) and \( i = 1, 2, \ldots, n \). It follows that
\[ z_j(t) = 0 \quad (i = 1, \ldots, m) \]
and
\[ z_j(t) = (\dot{z}_j - 1) \cdot z_j(t) \quad (j = m + 1, \ldots, n) \]
for \( t \geq 0 \). Then, we have
\[ z_i(t) = z_i(0) \quad (i = 1, \ldots, m) \]
and
\[ z_i(t) = z_i(0) \cdot e^{\lambda_{j-1} t} \quad (j = m + 1, \ldots, n) \]
for \( t \geq 0 \).

Let
\[ \lambda = \max_{m+1 \leq j \leq n} \{ \lambda_j \} < 1 \]
and
\[ -\varepsilon = \lambda - 1, \]
thus
\[ u(t) = \sum_{i=1}^{n} z_i(t) S_i = \sum_{j=m+1}^{n} z_j(0) S_j \cdot e^{-\lambda_{j-1} t} \leq \sum_{j=m+1}^{n} z_j(0) S_j \cdot e^{-\lambda_{j-1} t} \]
\[ = \sum_{i=1}^{n} z_i(0) S_i \cdot e^{-\varepsilon t} = (x(0, \phi) - x(0, \psi)) e^{-\varepsilon t} = (\phi - \psi) e^{-\varepsilon t} \]
for \( t \geq 0 \). Then
\[ ||x(t, \phi) - x(t, \psi)|| \leq ||\phi - \psi|| e^{-\varepsilon t} \]
for \( t \geq 0 \). By Lemma 2, the network (1) exists one periodic trajectory located in \( D \) and it exponentially attracts all trajectories in \( D \).

Given any initial point \( x(0) \), there is an invariant set \( D \) such that \( x(0) \in D \), so there are infinite invariant sets of (1) which are orthogonal to \( V_{d_i} \) in the state space, thus the linear network (1) has infinite exponentially stable periodic trajectories. The proof is complete. \( \square \)

This can be illustrated by the following 2-D network:
\[ \dot{x}(t) + x(t) = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} x(t) + \begin{bmatrix} \cos(t) \\ -\cos(t) \end{bmatrix} \]
for \( t \geq 0 \). The difference in this model and network (4) is the background. In this model the background \( b(t) \) is a periodic vector function.

The black line in Fig. 3 is the 1-D \( V_{d_i} \) in the 2-D space. The open circles are randomly selected initial points. The blue lines starting from these initial points are trajectories of the network. All the periodic trajectories are oscillating on the normal direction of \( V_{d_i} \) and through \( V_{d_i} \). The two red lines in Fig. 3 are the boundaries of the periodic trajectories. We can see that the center of every periodic solution is on \( V_{d_i} \). This figure is similar to Fig. 2 but they are different. The trajectories are attracted to the point on the continuous attractor in Fig. 2, while in Fig. 3 the trajectories are vibrating around the black line.

Let us consider a 3-D network:
\[ \dot{x}(t) + x(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} \cos(t) \\ -\cos(t) \\ -\cos(t) \end{bmatrix} \]
for \( t \geq 0 \).

Fig. 3. Periodic trajectories of model (6). The black line is \( V_{d_i} \). The open circles are randomly selected initial points. The blue lines starting from these initial points are trajectories of the network. All the periodic trajectories are oscillating back and forth on the normal direction of the black line and the center of every periodic solution is on the black line. The two red lines are the boundaries of the periodic trajectories. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Fig. 4. Periodic trajectories of model (7). The gray plane is \( V_{d_i} \). The open circles are 30 randomly selected initial points and the trajectories are lines oscillating on the normal direction of \( V_{d_i} \).
It can be checked that the largest eigenvalue is $\lambda_1 = 1$ with multiplicity 2, and another eigenvalue $\lambda_2 = -2$. The eigenvectors belong to the eigenvalue $\lambda_1 = 1$ are

$$S_1 = \begin{bmatrix} 0.3938 \\ 0.8163 \\ -0.4225 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0.7152 \\ 0.0166 \\ 0.6987 \end{bmatrix}.$$  

Clearly, $b(t) \in V_{h_1}$. By Theorem 3, the network possesses infinite periodic trajectories. The gray plane in Fig. 4 is the 2-D $V_{h_1}$ in the 3-D space. The open circles are 30 randomly selected initial points. The trajectories are also lines oscillating on the normal direction of $V_{h_1}$.

We can see that the center of every periodic solution is on this plane in Fig. 5.

The homogeneous form of network (1) is

$$x(t) + x(t) = Wx(t)$$

for $t \geq 0$.

Under the condition of Theorem 3, homogeneous network (8) has a continuous attractor. Actually the continuous attractor of (8) is $V_{h_1}$. The black line in Fig. 3 and the gray plane in Fig. 4 are the continuous attractors of the homogeneous form of the networks, respectively.

In the Fourier analysis, all periodic functions can be decomposed into the sum of a set of simple oscillating functions, namely sines and cosines. In order to understand the above theorem clearly, we investigate the simplest cases that the background inputs are cosine function and sine function.

**Theorem 4.** Suppose $b(t) = \cos(t)$, $\lambda_1 = 1$ is the largest eigenvalue of $W$ with the multiplicity $m$. Then, the periodic trajectories of linear network (1) can be represented by

$$P = \left\{ \sum_{i=1}^{m} (c_i + d_i \sin(t))S_i + \sum_{j=m+1}^{n} k_j(t)S_j \right\},$$

where

$$c_i \in \mathbb{R} \quad (1 \leq i \leq m),$$

$$k_j(t) = d_j \frac{1-\lambda_j \cos(t) + \sin(t)}{1+(1-\lambda_j)^2} \quad (m+1 \leq j \leq n).$$

Moreover, the periodic trajectories have the same period as the background input $b(t)$.

**Proof.** Since $b(t) = \cos(t)$, then $b_i(t) = d_i \cos(t)$, where $d_i \in \mathbb{R}$.

Since the multiplicity of the largest eigenvalue $\lambda_1 = 1$ is $m$, it follows from (1) and (2) that

$$z_i(t) = d_i \cos(t), \quad 1 \leq i \leq m$$

and

$$z_j(t) = (\lambda_j - 1)z_j(t) + d_j \cos(t), \quad m+1 \leq j \leq n$$

for $t \geq 0$.

It is easy to see that the derivative of $z_j(t)$ is zero on the tangent direction of $V_{h_1}$. Solving this equation, it gives that

$$z_j(t) = z_j(0) + d_j \sin(t), \quad 1 \leq i \leq m$$

and

$$z_j(t) = k_j(t) + (z_j(0) - \eta_j)e^{\lambda_j t} \quad m+1 \leq j \leq n$$

for $t \geq 0$, where

$$k_j(t) = d_j \frac{(1-\lambda_j \cos(t) + \sin(t))}{1+(1-\lambda_j)^2},$$

$$\eta_j = \frac{d_j (1-\lambda_j)}{1+(1-\lambda_j)^2}.$$ 

Thus

$$x(t) = \sum_{i=1}^{m} (z_i(0) + d_i \sin(t))S_i + \sum_{j=m+1}^{n} k_j(t)S_j + \sum_{j=m+1}^{n} (z_j(0) - \eta_j)e^{\lambda_j t}S_j,$$ \quad (9)

for $t \geq 0$.

Then we can get from (9) that when $t \to +\infty$, the third term on the right hand of (9)

$$\sum_{j=m+1}^{n} (z_j(0) - \eta_j)e^{\lambda_j t}S_j \to 0,$$

so when $t \to +\infty$,

$$x(t) \to \sum_{i=1}^{m} (z_i(0) + d_i \sin(t))S_i + \sum_{j=m+1}^{n} k_j(t)S_j.$$ \quad \square

**Proof.** We choose different initial point $x(0)$, then the continuous periodic attractor is

$$P = \left\{ \sum_{i=1}^{m} (c_i + d_i \sin(t))S_i + \sum_{j=m+1}^{n} k_j(t)S_j \right\},$$

where

$$c_i \in \mathbb{R} \quad (1 \leq i \leq m),$$

$$k_j(t) = d_j \frac{(1-\lambda_j \cos(t) + \sin(t))}{1+(1-\lambda_j)^2} \quad (m+1 \leq j \leq n).$$

From the form of $c_i + d_i \sin(t)$ and $k_j(t)$ we can see that the period of $P$ is as same as the background input. The proof is complete. \quad \square

**Corollary 1.** Suppose $b(t) = \sin(t)$, $\lambda_1 = 1$ is the largest eigenvalue of $W$ with the multiplicity $m$. Then, the periodic trajectories of linear network (1) can be represented by

$$P = \left\{ \sum_{i=1}^{m} (c_i + d_i (1-\cos(t)))S_i + \sum_{j=m+1}^{n} k_j(t)S_j \right\},$$

where

$$c_i \in \mathbb{R} \quad (1 \leq i \leq m),$$
Theorem 4 and Corollary 1, the background input is very different from the one with constant background. In clear representation of the stable periodic trajectories of linear networks with any periodic background input.

In fact, the analysis of network with periodic background input is very different from the one with constant background. In Theorem 4 and Corollary 1, the background input \( \cos(t) \) and \( \sin(t) \) are not orthogonal to \( V_{ji} \), and the stable periodic trajectories can be obtained in the same way. So whether the background input is orthogonal to \( V_{ji} \), or not, the network possesses stable periodic trajectories.

Let us see some examples:

Consider a two-dimensional network

\[
\dot{x}(t) + x(t) = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} x(t) + \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}
\]

(10)

for \( t \geq 0 \). Here the background is not orthogonal to \( V_{ji} \).

The network possesses infinite periodic trajectories. In order to see clearly, only five trajectories are plotted. The curves in different colors are stable periodic trajectories of the network. The black line in Fig. 6 is the 1-D \( V_{ji} \) in the 2-D space. All the periodic trajectories are oscillating back and forth and orthogonal to \( V_{ji} \). We can see that the center of every periodic solution is on \( V_{ji} \).

Consider another 3-D network

\[
\dot{x}(t) + x(t) = \begin{bmatrix} -1 & -2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} \cos(t) \\ \cos(t) \\ \cos(t) \end{bmatrix}
\]

(11)

for \( t \geq 0 \). Clearly

\[
b(t) = \begin{bmatrix} \cos(t) \\ \cos(t) \\ \cos(t) \end{bmatrix}
\]

is periodic vector function. It can be checked that the \( W \) has the largest \( \lambda_1 = 1 \) with multiplicity 1, and \( W \) has another eigenvalue \( \lambda_2 = 0 \). The eigenvector of \( W \) belong to the eigenvalue \( \lambda_1 = 1 \) is

\[
S_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}
\]

By Theorem 4, the network possesses infinite periodic trajectories. The black line in Fig. 7 is the 1-D \( V_{ji} \) in the 3-D space. The open circles are five randomly selected initial points. The curves in different colors starting from these initial points are stable periodic trajectories of the network. All the periodic trajectories are oscillating back and forth and orthogonal to \( V_{ji} \). The center of every periodic solution is on \( V_{ji} \).

Although all the periodic trajectories look like being on a cylinder, they are located on a plane. We can see it clearly in Fig. 8.

![Fig. 6. Periodic trajectories of model (10). The black line is \( V_{ji} \). The open circles are five randomly selected initial points. The curves in different colors starting from these initial points are stable periodic trajectories of the network. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)](image_url)

![Fig. 7. Periodic trajectories of model (11). The black line is \( V_{ji} \) in the 3-D space. The open circles are five randomly selected initial points. The curves in different colors starting from these initial points are stable periodic trajectories of the network. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)](image_url)

![Fig. 8. Periodic trajectories of model (11) from another view. All the periodic trajectories of model (11) are located on a plane.](image_url)
Consider another 3-D network:

$$x(t) + x(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} \cos(t) \\ \cos(t) \\ \cos(t) \end{bmatrix}$$

for $t \geq 0$. Clearly

$$b(t) = \begin{bmatrix} \cos(t) \\ \cos(t) \\ \cos(t) \end{bmatrix}$$

By Theorem 4, the network possesses infinite stable periodic trajectories. Each trajectory is an ellipse. The gray plane in Fig. 9 is the 2-D $V_{\lambda_2}$, $V_{\lambda_3}$ is combined by the two eigenvectors of connection matrix associated with the largest eigenvalue 1. Five trajectories starting from the initial points are oscillating back and forth on the normal direction of $V_{\lambda_2}$. It is interesting that the focuses of every ellipse trajectory are on $V_{\lambda_2}$ in Fig. 10.

In all the above examples the periodic trajectory may be line or ellipse and all the trajectories may be located in a part of 2-D state space, a low-dimensional plane in 3-D space or the whole high-dimensional state space. There may be also other kinds of trajectories. What unchangeable is that all the trajectories are located in the invariant set which is orthogonal to $V_{\lambda_2}$, $V_{\lambda_3}$ is dependent on the connection matrix $W$. The connection matrix and the background input decide the periodic trajectories form together.

Some basic theories of continuous attractor neural networks with oscillating background tuning have been investigated in this section. One example of the application of this theory is manifold learning. Recently, manifold learning is a popular recent approach to find a low-dimensional basis for describing high-dimensional data. Seung and Lee have shown in their paper [16] that as the faces are rotated, they trace out continuous curves embedded in image space. These curves are the continuous periodic trajectories and the rotated faces are the oscillating background inputs. These curves are continuous because the images vary smoothly as the faces are rotated. They are curves because they are generated by varying a single degree of freedom, the angle of rotation. Moreover, these curves are low dimensional, although they are embedded in image space, which has a high dimensionality equal to the number of image pixels. In fact, memories of the face patterns are stored in low-dimensional continuous curves. The connections between such neural manifolds and the image manifolds help us to understand the memory storage in the brain. More applications about this theory may be discovered gradually in the future.

5. Conclusions

In this work we studied the dynamic properties of continuous attractor neural networks with background tuning. We used a linear neural network to demonstrate that our neural network model can be programmed to relax to appropriate initial conditions to continuous attractors. By tuning background input the continuous attractor shift scheme was studied. Two background tuning mechanisms were investigated in detail. First, when the background is set to be constant and changed smoothly, the continuous attractor is also shifted continuously to another one. Second, the explicit representations of periodic trajectories are given with periodic oscillation background. The continuous attractor neural network is understood deeply. The method given in this paper may be further developed to other network models.

References


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