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Adaptive Control for Parametric Output Feedback Systems with Output Constraint

Beibei Ren, Shuzhi Sam Ge*, Keng Peng Tee and Tong Heng Lee

Abstract— **In this paper, adaptive control is presented for a class of parametric output feedback nonlinear systems with output constraint. Adaptive observer backstepping is adopted to achieve the output tracking. To prevent the output constraint violation, the Barrier Lyapunov Function (BLF) is employed in Lyapunov synthesis. By ensuring the boundedness of the BLF, we also guarantee that the output constraint is not transgressed. Compared with the control using a Quadratic Lyapunov Function (QLF), our control requires less restrictive initial conditions. The stability analysis of the closed-loop system is provided and all closed-loop signals are ensured to be bounded. Simulation results demonstrate the effectiveness of the proposed approach.**

I. Introduction

Constraints are ubiquitous in physical systems, and manifest themselves as physical stoppages, saturation, as well as performance and safety specifications. Violation of the constraints during operation may result in performance degradation, hazards or system damage. Driven by practical needs and theoretical challenges, the rigorous handling of constraints in control design has become an important research topic in recent decades. Existing methods to handle constraints include artificial potential field [1], model predictive control [2], [3], reference governors [4], [5], the use of set invariance notions [6], [7].

Different from the above-mentioned methods, one can use Barrier Lyapunov Functions (BLFs) to tackle the issue of constraint, which avoids the need for explicit solutions of the system by virtue of being a Lyapunov based control design methodology. For the great majority of works in the literature, the constructed Lyapunov functions are radially unbounded, for global stability, or at least well-defined over the entire domain. In contrast to this convention, the BLF-based method exploits the property that the value of the barrier function approaches infinity whenever its arguments approach certain limits. The design of barrier functions in Lyapunov synthesis has been proposed for constraint handling in Brunovsky-type systems [8]. Inspired by [8], our previous work [9] has proposed novel asymmetric BLFs and presented both symmetric and asymmetric BLFs control design for single-input single-output (SISO) nonlinear systems in strict feedback form with an output constraint and parametric uncertainties.

To the best of the authors' knowledge, there has been relatively few works in the literature on the control problem of output feedback nonlinear systems with constraints. Designing an output feedback control for nonlinear systems with guarantee of constraint satisfaction is still an open and challenging problem. A natural and promising point to start is with nonlinear systems in the output feedback form under output constraint, since such systems are amenable to adaptive observer backstepping techniques that can be fused with BLFs. In our previous work [10], asymmetric BLF was employed for the control of electrostatic parallel plate microactuators with guaranteed non-contact between the movable and fixed electrodes, where the plant can be transformed into the second-order parametric output feedback form. In this paper, we extend the results in [10] to a more general class of parametric output feedback nonlinear systems with output constraint. Following the constructive procedures of adaptive observer backstepping design in [11], we incorporate the BLF into Lyapunov synthesis to prevent the output constraint violation. The main feature of our method is that BLFs may yield less restrictive requirements on the initial conditions than Quadratic Lyapunov Functions (QLFs).

The organization of this paper is as follows. The problem formulation and preliminaries are given in Section II. Section III presents the state estimation filter and observer design. In Section IV, the constructive procedures of adaptive observer backstepping design are provided and the closed-loop system stability is analyzed as well. Section V demonstrates the feasibility of the proposed approach using a numerical example. Conclusions follow in Section VI.

II. Problem Formulation and Preliminaries

A. Problem Formulation

Consider the following output feedback nonlinear system, whose nonlinearities depend only on the output y:

$$\begin{aligned}
\dot{x}_1 &= x_2 + \phi_1^T(y)\theta + \psi_1(y) \\
&\vdots \\
\dot{x}_{\rho-1} &= x_\rho + \phi_{\rho-1}^T(y)\theta + \psi_{\rho-1}(y) \\
\dot{x}_\rho &= x_{\rho+1} + \phi_\rho^T(y)\theta + \psi_\rho(y) + b_m u \\
&\vdots \\
\dot{x}_{n-1} &= x_n + \phi_{n-1}^T(y)\theta + \psi_{n-1}(y) + b_1 u \\
\dot{x}_n &= \phi_n^T(y)\theta + \psi_n(y) + b_0 u \\
y &= x_1
\end{aligned} \tag{1}$$

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where $x_1, ..., x_n$ are system states, y and u are the output and input respectively; $\psi_i(y)$ and $\phi_i(y) \in R^r$, $i = 1, ..., n$ are known smooth functions; $\theta \in R^r$ is a vector of uncertain constant parameters satisfying $\|\theta\| \leq \theta_M$ with known positive constant θ_M; and $b_m, ..., b_0$ are uncertain constant parameters satisfying that $B_N \leq |b_i| \leq B_M$, $i = 0, 1, ..., m$ with known positive constants B_N and B_M. We consider the problem of output constraint, for which the output is required to remain in the set $|y| \leq k_{c_1}$, where k_{c_1} is a positive constant.

The following assumptions are made for the system (1):

Assumption 1: The sign of b_m is known.

Assumption 2: The relative degree $\rho = n - m$ is known and the system is minimum phase.

Assumption 3: For any $k_{c_1} > 0$, there exist positive constants $\underline{Y}_0, \bar{Y}_0, A_0, Y_1, Y_2, ..., Y_n$ satisfying $\max\{\underline{Y}_0, \bar{Y}_0\} \leq A_0 < k_{c_1}$ such that the reference signal $y_r(t)$ and its ρth order derivatives are piecewise continuous, known and bounded, which satisfy $-\underline{Y}_0 \leq y_r(t) \leq \bar{Y}_0$, $|\dot{y}_r(t)| < Y_1$, $|\ddot{y}_r(t)| < Y_2$, ..., $|y_r^{(n)}(t)| < Y_n$, $\forall t \geq 0$.

The reference signal $y_r(t)$, and its ρth order derivatives are piecewise continuous, known and bounded.

Assuming that only the output signal y is measured, the control objective is to track the given reference signal $y_r(t)$ with the output y, while keeping that all of the signals in the closed-loop system bounded and that the output constraint is not violated.

B. Barrier Lyapunov Function

To prevent the output from violating the constraint, we provide the definition of Barrier Lyapunov Function and one useful lemma as follows.

Definition 1: [9] A Barrier Lyapunov Function (BLF) is a scalar function $V(x)$, defined with respect to the system $\dot{x} = f(x)$ on an open region $\mathcal{D}$ containing the origin, that is continuous, positive definite, has continuous first-order partial derivatives at every point of $\mathcal{D}$, has the property $V(x) \to \infty$ as x approaches the boundary of $\mathcal{D}$, and satisfies $V(x(t)) \leq b$ $\forall t \geq 0$ along the solution of $\dot{x} = f(x)$ for $x(0) \in \mathcal{D}$ and some positive constant b.

Lemma 1: [9] Let $z_1(t), z_2(t)$ be continuously differentiable trajectories with initial conditions $z_1(0) \in (-k_{a_1}, k_{b_1})$, $z_2(0) \in R^r$, where k_{a_1} and k_{b_1} are positive constants. If there exists a continuously differentiable and positive definite function

$$V(z_1, z_2) = V_1(z_1) + V_2(z_2)$$

defined on $z_1 \in (-k_{a_1}, k_{b_1})$, $z_2 \in \mathcal{R}^r$, and class $\mathcal{K}_\infty$ functions γ_1 and γ_2, such that (i) $V_1(z_1) \to \infty$ as $z_1 \to -k_{a_1}$ or $z_1 \to k_{b_1}$, (ii) $\gamma_1(\|z_2\|) \leq V_2(z_2) \leq \gamma_2(\|z_2\|)$, and (iii)$\dot{V} \leq 0$, then $z_1(t)$ remains in the set $z_1 \in (-k_{a_1}, k_{b_1})$, $\forall t < 0$.

As discussed in [9], there are many functions satisfying Definition 1, which may be symmetric or asymmetric as illustrated in Figure 1. For clarity, the following symmetric BLF candidate considered in [8][9] is used throughout this paper:

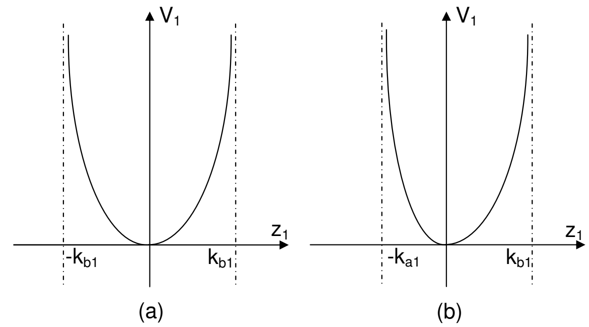

Fig. 1. Schematic illustration of (a) symmetric and (b) asymmetric barrier functions

$$V_1 = \frac{1}{2} \log \frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2} \tag{2}$$

where $\log(\cdot)$ denotes the natural logarithm of $\cdot$, and k_{b_1} the constraint on z_1, i.e., $|z_1| < k_{b_1}$. As seen from the schematic illustration of $V_1(z_1)$ in Figure 1 (a), the BLF escapes to infinity at $|z_1| = k_{b_1}$. It can be shown that V_1 is positive definite and C^1 continuous in the set $|z_1| < k_{b_1}$, and thus a valid Lyapunov function candidate. The control design in this paper can be easily extended to the asymmetric BLF case. Interested readers can refer to [9].

Throughout this paper, we denote $\bar{z}_i = [z_1, z_2, ..., z_i]^T$, $\bar{\lambda}_i = [\lambda_1, \lambda_2, ..., \lambda_i]^T$ and $\bar{y}_r^{(i)} = [y_r, y_r^{(1)}, ..., y_r^{(i)}]^T$.

III. State Estimation Filter and Observer Design

Since only the output signal y is measured, some filters should be designed first which will provide "virtual estimates" of the unmeasured state variables $x_2, ..., x_n$. We rewrite the plant (1) in the following form

$$\dot{x} = Ax + \Phi(y)\theta + \Psi(y) + \begin{bmatrix} 0 \\ b \end{bmatrix} u \tag{3}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad \Psi(y) = \begin{bmatrix} \psi_1(y) \\ \vdots \\ \psi_n(y) \end{bmatrix}$$

$$\Phi(y) = \begin{bmatrix} \phi_1^T(y) \\ \vdots \\ \phi_n^T(y) \end{bmatrix} \quad b = \begin{bmatrix} b_m \\ \vdots \\ b_0 \end{bmatrix}$$

Choose the K-filters [11] as follows:

$$\dot{\xi} = A_0\xi + ky + \Psi(y) \tag{4}$$

$$\dot{\Xi}^T = A_0\Xi^T + \Phi(y) \tag{5}$$

$$\dot{\lambda} = A_0\lambda + e_n u \tag{6}$$

$$v_i = A_0^i\lambda, \ i = 0, 1, ..., m \tag{7}$$

where $k = [k_1, ..., k_n]^T$ such that $A_0 = A - ke_1^T$ is Hurwitz, and e_i is the ith coordinate vector.

By constructing the state estimates as follows:

$$\hat{x}(t) = \xi + \Xi^T\theta + \sum_0^m b_i v_i \tag{8}$$

it is straightforward to verify that the dynamics of the observation error, $\varepsilon = x - \hat{x}$, are given by

$$\dot{\varepsilon} = A_0\varepsilon \tag{9}$$

Since A_0 is Hurwitz, (9) implies that ε converges exponentially to zero. Furthermore, the dynamic equation of y can be expressed as

$$\begin{aligned} \dot{y} &= x_2 + \phi_1^T(y)\theta + \psi_1(y) \\ &= b_m v_{m,2} + \xi_2 + \psi_1(y) + \bar{\Omega}^T\Theta + \varepsilon_2 \end{aligned} \tag{10}$$

with

$$\begin{aligned} \Theta &= [b_m, ..., b_0, \theta^T]^T \\ \Omega &= [v_{m,2}, v_{m-1,2}, ..., v_{0,2}, \Xi_2 + \phi_1^T]^T \\ \bar{\Omega} &= [0, v_{m-1,2}, ..., v_{0,2}, \Xi_2 + \phi_1^T]^T \end{aligned}$$

where ε_2, $v_{i,2}$, ξ_2 and Ξ_2 denote the second entries of ε, v_i, ξ and Ξ, respectively, and y, v_i, ξ and Ξ are all available signals. It is obvious that there exists some positive constant Θ_M, such that $\|\Theta\| \leq \Theta_M$.

Combining system (10) with the filters (4)-(7), system (1) is represented as

$$\begin{aligned} \dot{y} &= b_m v_{m,2} + \xi_2 + \psi_1(y) + \bar{\Omega}^T\Theta + \varepsilon_2 && (11) \\ \dot{v}_{m,i} &= v_{m,i+1} - k_i v_{m,1}, \quad i = 2, 3, ..., \rho - 1 && (12) \\ \dot{v}_{m,\rho} &= v_{m,\rho+1} - k_\rho v_{m,1} + u && (13) \end{aligned}$$

In the next section, adaptive observer backstepping design will be presented for the system (11)-(13) with constructive procedures, where states y, $v_{m,2}$, ..., $v_{m,\rho}$ are available.

IV. Adaptive Observer Backstepping Design and Stability Analysis

In this section, we present the adaptive control design using the backstepping technique with tuning functions in ρ steps. Define the following the error coordinates:

$$\begin{aligned} z_1 &= y - y_r && (14) \\ z_i &= v_{m,i} - \alpha_{i-1} - \hat{\varrho} y_r^{(i-1)}, \ i = 2, 3, ..., \rho && (15) \end{aligned}$$

where $\hat{\varrho}$ is an estimate of $\varrho = \frac{1}{b_m}$ and α_{i-1} is the stabilizing functions at each step and will be defined later.

Step 1: From (11) and (14), the derivative of z_1 is given by

$$\dot{z}_1 = b_m v_{m,2} + \xi_2 + \psi_1(y) + \bar{\Omega}^T\Theta + \varepsilon_2 - \dot{y}_r \tag{16}$$

By substituting (15) for $i = 2$ into (16) and using $\tilde{\varrho} = \frac{1}{b_m} - \frac{1}{\hat{b}_m}$, we have

$$\begin{aligned} \dot{z}_1 &= b_m\alpha_1 + \xi_2 + \psi_1(y) + \bar{\Omega}^T\Theta + \varepsilon_2 - b_m\tilde{\varrho}\dot{y}_r \\ &\quad + b_m z_2 \end{aligned} \tag{17}$$

To design a control that does not drive y out of the interval $|y| < k_{c_1}$, which implies that $|z_1| < k_{b_1}$ with $k_{b_1} = k_{c_1} - A_0$, we choose the following Lyapunov function candidate, which incorporate the symmetric Barrier Lyapunov Function candidate introduced in Section II-B:

$$\begin{aligned} V_1 &= \frac{1}{2}\log\frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2} + \frac{1}{2}\tilde{\Theta}^T\Gamma^{-1}\tilde{\Theta} + \frac{|b_m|}{2\gamma_\varrho}\tilde{\varrho}^2 \\ &\quad + \frac{1}{2d_1}\varepsilon^T P\varepsilon \end{aligned} \tag{18}$$

where $\tilde{\Theta} = \Theta - \hat{\Theta}$, Γ is a positive definite design matrix, γ_ϱ and d_1 are positive design parameters, and P is a definite positive matrix such that $PA_0 + A_0^T P = -I$, $P = P^T > 0$. The derivative of V_1 along (17) is given by

$$\begin{aligned} \dot{V}_1 &= \frac{z_1\dot{z}_1}{k_{b_1}^2 - z_1^2} - \tilde{\Theta}^T\Gamma^{-1}\dot{\hat{\Theta}} - \frac{|b_m|}{\gamma_\varrho}\tilde{\varrho}\dot{\hat{\varrho}} - \frac{1}{2d_1}\varepsilon^T\varepsilon \\ &= \frac{z_1}{k_{b_1}^2 - z_1^2}[b_m\alpha_1 + \xi_2 + \psi_1(y) + \bar{\Omega}^T\Theta + \varepsilon_2 \\ &\quad - b_m\tilde{\varrho}\dot{y}_r + b_m z_2] - \tilde{\Theta}^T\Gamma^{-1}\dot{\hat{\Theta}} - \frac{|b_m|}{\gamma_\varrho}\tilde{\varrho}\dot{\hat{\varrho}} \\ &\quad - \frac{1}{2d_1}\varepsilon^T\varepsilon \end{aligned} \tag{19}$$

Design the following stabilizing functions:

$$\begin{aligned} \alpha_1 &= \hat{\varrho}\bar{\alpha}_1 && (20) \\ \bar{\alpha}_1 &= -c_1(k_{b_1}^2 - z_1^2)z_1 - \xi_2 - \psi_1(y) - \bar{\Omega}^T\hat{\Theta} \\ &\quad - \frac{d_1 z_1}{k_{b_1}^2 - z_1^2} && (21) \end{aligned}$$

where c_1 is a positive design parameter, and $\hat{\Theta}$ is the estimate of Θ.

It is easy to know that

$$b_m\alpha_1 = b_m\hat{\varrho}\bar{\alpha}_1 = \bar{\alpha}_1 - b_m\tilde{\varrho}\bar{\alpha}_1 \tag{22}$$

Substituting (20)-(22) into (19) leads to

$$\begin{aligned} \dot{V}_1 &= -c_1 z_1^2 + \frac{z_1}{k_{b_1}^2 - z_1^2}(\bar{\Omega}^T\tilde{\Theta} + b_m z_2) \\ &\quad - \frac{z_1}{k_{b_1}^2 - z_1^2} b_m\tilde{\varrho}(\dot{y}_r + \bar{\alpha}_1) + \frac{z_1\varepsilon_2}{k_{b_1}^2 - z_1^2} \\ &\quad - d_1\Big(\frac{z_1}{k_{b_1}^2 - z_1^2}\Big)^2 - \tilde{\Theta}^T\Gamma^{-1}\dot{\hat{\Theta}} - \frac{|b_m|}{\gamma_\varrho}\tilde{\varrho}\dot{\hat{\varrho}} \\ &\quad - \frac{1}{2d_1}\varepsilon^T\varepsilon \end{aligned} \tag{23}$$

For the second term in the right hand of (23), we can rewrite it as

$$\begin{aligned} &\frac{z_1}{k_{b_1}^2 - z_1^2}(\bar{\Omega}^T\tilde{\Theta} + b_m z_2) \\ = &\frac{z_1}{k_{b_1}^2 - z_1^2}[\bar{\Omega}^T\tilde{\Theta} + (v_{m,2} - \hat{\varrho}\dot{y}_r - \alpha_1)e_1^T\tilde{\Theta} + \hat{b}_m z_2] \\ = &\frac{z_1}{k_{b_1}^2 - z_1^2}\{[\Omega - \hat{\varrho}(\dot{y}_r + \bar{\alpha}_1)e_1]^T\tilde{\Theta} + \hat{b}_m z_2\} \end{aligned} \tag{24}$$

By using Young's inequality, the fourth term in the right hand of (23) becomes

$$\begin{aligned} \frac{z_1\varepsilon_2}{k_{b_1}^2 - z_1^2} &\leq d_1\Big(\frac{z_1}{k_{b_1}^2 - z_1^2}\Big)^2 + \frac{\varepsilon_2^2}{4d_1} \\ &\leq d_1\Big(\frac{z_1}{k_{b_1}^2 - z_1^2}\Big)^2 + \frac{1}{4d_1}\varepsilon^T\varepsilon \end{aligned} \tag{25}$$

Substituting (24) and (25) into (23), we have

$$\begin{aligned}\dot{V}_1 \;\le\;& -c_1 z_1^2 + \frac{\hat{b}_m z_1 z_2}{k_{b_1}^2 - z_1^2} \\ &+\tilde{\Theta}^T\Big\{\frac{z_1}{k_{b_1}^2 - z_1^2}[\Omega - \hat{\varrho}(\dot{y}_r + \bar{\alpha}_1)e_1] - \Gamma^{-1}\dot{\hat{\Theta}}\Big\} \\ &-\frac{|b_m|}{\gamma_\varrho}\tilde{\varrho}\Big[\gamma_\varrho \mathrm{sign}(b_m)(\dot{y}_r + \bar{\alpha}_1)\frac{z_1}{k_{b_1}^2 - z_1^2} + \dot{\hat{\varrho}}\Big] \\ &-\frac{1}{4d_1}\varepsilon^T\varepsilon \end{aligned} \tag{26}$$

Choose the adaptation law $\dot{\hat{\varrho}}$ and the tuning function τ_1 as follows:

$$\dot{\hat{\varrho}} = -\gamma_\varrho \mathrm{sign}(b_m)(\dot{y}_r + \bar{\alpha}_1)\frac{z_1}{k_{b_1}^2 - z_1^2} \tag{27}$$

$$\tau_1 = \frac{z_1}{k_{b_1}^2 - z_1^2}[\Omega - \hat{\varrho}(\dot{y}_r + \bar{\alpha}_1)e_1] \tag{28}$$

Substituting (27) and (28) into (26) results in

$$\begin{aligned}\dot{V}_1 \;\le\;& -c_1 z_1^2 + \frac{\hat{b}_m z_1 z_2}{k_{b_1}^2 - z_1^2} + \tilde{\Theta}^T(\tau_1 - \Gamma^{-1}\dot{\hat{\Theta}}) \\ &-\frac{1}{4d_1}\varepsilon^T\varepsilon \end{aligned} \tag{29}$$

with the coupling term $\frac{\hat{b}_m z_1 z_2}{k_{b_1}^2 - z_1^2}$ to be canceled in the subsequent step.

Step 2: The derivative of z_2 can be obtained from (12) and (15) as follows

$$\begin{aligned}\dot{z}_2 &= \dot{v}_{m,2} - \dot{\alpha}_1 - \dot{\hat{\varrho}}\dot{y}_r - \hat{\varrho}\ddot{y}_r \\ &= v_{m,3} - k_2 v_{m,1} - \dot{\alpha}_1 - \dot{\hat{\varrho}}\dot{y}_r - \hat{\varrho}\ddot{y}_r \end{aligned} \tag{30}$$

From (20) and (21), we know that α_1 is a function of $y, \xi, \Xi, \hat{\Theta}, \hat{\varrho}, y_r, \bar{\lambda}_{m+1}$, thus, its derivative $\dot{\alpha}_1$ can be expressed as

$$\begin{aligned}\dot{\alpha}_1 =\;& \frac{\partial \alpha_1}{\partial y}(b_m v_{m,2} + \xi_2 + \psi_1(y) + \bar{\Omega}^T\Theta + \varepsilon_2) + \frac{\partial \alpha_1}{\partial y_r}\dot{y}_r \\ &+\sum_{j=1}^{m+1}\frac{\partial \alpha_1}{\partial \lambda_j}(-k_j\lambda_1 + \lambda_{j+1}) + \frac{\partial \alpha_1}{\partial \xi}(A_0\xi + ky \\ &+\Psi(y)) + \frac{\partial \alpha_1}{\partial \Xi}(A_0\Xi^T + \Phi(y)) + \frac{\partial \alpha_1}{\partial \hat{\Theta}}\dot{\hat{\Theta}} + \frac{\partial \alpha_1}{\partial \hat{\varrho}}\dot{\hat{\varrho}} \end{aligned} \tag{31}$$

Substituting (31) into (30), we have

$$\begin{aligned}\dot{z}_2 =\;& v_{m,3} - \hat{\varrho}\ddot{y}_r - \beta_2 - \frac{\partial \alpha_1}{\partial y}(\Omega^T\tilde{\Theta} + \varepsilon_2) \\ &-\frac{\partial \alpha_1}{\partial \hat{\Theta}}\dot{\hat{\Theta}} \end{aligned} \tag{32}$$

where

$$\begin{aligned}\beta_2 =\;& \frac{\partial \alpha_1}{\partial y}(\xi_2 + \psi_1(y) + \Omega^T\hat{\Theta}) + k_2 v_{m,1} + \frac{\partial \alpha_1}{\partial y_r}\dot{y}_r \\ &+(\frac{\partial \alpha_1}{\partial \hat{\varrho}} + \dot{y}_r)\dot{\hat{\varrho}} + \sum_{j=1}^{m+1}\frac{\partial \alpha_1}{\partial \lambda_j}(-k_j\lambda_1 + \lambda_{j+1}) \\ &+\frac{\partial \alpha_1}{\partial \xi}(A_0\xi + ky + \Psi(y)) + \frac{\partial \alpha_1}{\partial \Xi}(A_0\Xi^T + \Phi(y)) \end{aligned} \tag{33}$$

Taking $v_{m,3}$ as a virtual control input and using $z_3 = v_{m,3} - \alpha_2 - \hat{\varrho}\ddot{y}_r$, we have

$$\dot{z}_2 = z_3 + \alpha_2 - \beta_2 - \frac{\partial \alpha_1}{\partial y}(\Omega^T\tilde{\Theta} + \varepsilon_2) - \frac{\partial \alpha_1}{\partial \hat{\Theta}}\dot{\hat{\Theta}} \tag{34}$$

Since z_2 does not need to be constrained, we choose Lyapunov function candidate by augmenting V_1 with a quadratic function as follows

$$V_2 = V_1 + \frac{1}{2}z_2^2 + \frac{1}{2d_2}\varepsilon^T P\varepsilon \tag{35}$$

where d_2 is a positive design parameter. The derivative of V_2 along (29) and (34) is given by

$$\begin{aligned}\dot{V}_2 \;\le\;& -c_1 z_1^2 + \frac{\hat{b}_m z_1 z_2}{k_{b_1}^2 - z_1^2} + z_2\Big[z_3 + \alpha_2 - \beta_2 \\ &-\frac{\partial \alpha_1}{\partial y}(\Omega^T\tilde{\Theta} + \varepsilon_2) - \frac{\partial \alpha_1}{\partial \hat{\Theta}}\dot{\hat{\Theta}}\Big] \\ &-\frac{1}{2d_2}\varepsilon^T\varepsilon - \frac{1}{4d_1}\varepsilon^T\varepsilon + \tilde{\Theta}^T(\tau_1 - \Gamma^{-1}\dot{\hat{\Theta}}) \end{aligned} \tag{36}$$

Choose the second stabilizing function α_2 and tuning function τ_2:

$$\begin{aligned}\alpha_2 =\;& -\frac{\hat{b}_m z_1}{k_{b_1}^2 - z_1^2} - c_2 z_2 + \beta_2 + \frac{\partial \alpha_1}{\partial \hat{\Theta}}\Gamma\tau_2 \\ &-d_2\Big(\frac{\partial \alpha_1}{\partial y}\Big)^2 z_2 \end{aligned} \tag{37}$$

$$\tau_2 = \tau_1 - \frac{\partial \alpha_1}{\partial y}\Omega z_2 \tag{38}$$

where c_2 is a positive design parameter. Substituting (37) and (38) into (39), we have

$$\begin{aligned}\dot{V}_2 \;\le\;& -\sum_{i=1}^{2}\Big(c_i z_i^2 + \frac{1}{4d_i}\varepsilon^T\varepsilon\Big) + z_2 z_3 + \tilde{\Theta}^T(\tau_2 - \Gamma^{-1}\dot{\hat{\Theta}}) \\ &+z_2\frac{\partial \alpha_1}{\partial \hat{\Theta}}(\Gamma\tau_2 - \dot{\hat{\Theta}}) \end{aligned} \tag{39}$$

Step $i = 3, ..., \rho$**.** Similar to the procedures in Step 2, consider the Lyapunov function candidates:

$$V_i = V_{i-1} + \frac{1}{2}z_i^2 + \frac{1}{2d_i}\varepsilon^T P\varepsilon, \quad i = 3, ..., \rho \tag{40}$$

and choose the following stabilizing functions α_i and tuning functions τ_i for $i = 3, ..., \rho$:

$$\begin{aligned}\alpha_i =\;& -z_{i-1} - c_i z_i + \beta_i + \frac{\partial \alpha_{i-1}}{\partial \hat{\Theta}}\Gamma\tau_i - d_i\Big(\frac{\partial \alpha_{i-1}}{\partial y}\Big)^2 z_i \\ &-\Big(\sum_{k=2}^{i-1} z_k\frac{\partial \alpha_{k-1}}{\partial \hat{\Theta}}\Big)\Gamma\frac{\partial \alpha_{i-1}}{\partial y}\Omega \end{aligned} \tag{41}$$

$$\tau_i = \tau_{i-1} - \frac{\partial \alpha_{i-1}}{\partial y}\Omega z_i \tag{42}$$

where

$$\beta_i = \frac{\partial\alpha_{i-1}}{\partial y}(\xi_2 + \psi_1(y) + \Omega^T\hat{\Theta}) + k_i v_{m,1} + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_r^{(j-1)}}\dot{y}_r^{(j)} + (\frac{\partial\alpha_{i-1}}{\partial\hat{\varrho}} + \dot{y}_r^{(i-1)})\dot{\hat{\varrho}} + \sum_{j=1}^{m+i-1}\frac{\partial\alpha_{i-1}}{\partial\lambda_j}(-k_j\lambda_1 + \lambda_{j+1}) + \frac{\partial\alpha_{i-1}}{\partial\xi}(A_0\xi + ky + \Psi(y)) + \frac{\partial\alpha_{i-1}}{\partial\Xi}(A_0\Xi^T + \Phi(y)) \quad (43)$$

In the last step ρ, the actual control law u and the adaptation law $\dot{\hat{\Theta}}$ are given as follows:

$$u = \alpha_\rho - v_{m,\rho+1} + \hat{\varrho}y_r^{(\rho)} \quad (44)$$

$$\dot{\hat{\Theta}} = \Gamma\tau_\rho \quad (45)$$

The final Lyapunov function V_ρ can be written as

$$V_\rho = \frac{1}{2}\log\frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2} + \sum_{i=2}^{\rho}\frac{1}{2}z_i^2 + \frac{1}{2}\tilde{\Theta}^T\Gamma^{-1}\tilde{\Theta} + \frac{|b_m|}{2\gamma_\varrho}\tilde{\varrho}^2 + \sum_{i=1}^{\rho}\frac{1}{2d_i}\varepsilon^T P\varepsilon \quad (46)$$

Noting that

$$\Gamma\tau_{i-1} - \dot{\hat{\Theta}} = \Gamma\tau_{i-1} - \Gamma\tau_i + \Gamma\tau_i - \dot{\hat{\Theta}} = \Gamma\frac{\partial\alpha_{i-1}}{\partial y}\Omega z_i \quad (47)$$

and substituting (39)-(45) into the derivative of V_ρ, we obtain

$$\dot{V}_\rho \le -\sum_{i=1}^{\rho}c_i z_i^2 - \sum_{i=1}^{\rho}\frac{1}{4d_i}\varepsilon^T\varepsilon \quad (48)$$

Theorem 1: Consider the closed-loop system consisting of the plant (1), filters (4)-(7), stabilizing functions (20)(41), control law (44) and adaptation laws (27)(45), under Assumptions 1-3. If the initial conditions start from $\bar{z}_n(0) \in \Omega_{z_0} := \{\bar{z}_n \in R^n : |z_1| < k_{b_1}\}$, then the following properties hold:

(i) the output constraint is never violated, i.e., $|y(t)| < k_{c_1}$, $\forall t > 0$;
(ii) all closed loop signals are bounded;
(iii) the asymptotic tracking is achieved, i.e., $y(t) \to y_r(t)$ as $t \to \infty$.

Proof:

(i) From (48), we know that $\dot{V}_\rho \le 0$, which leads to $V_n(t) \le V_n(0)$. According to Lemma 1, if $|z_1(0)| < k_{b_1}$, we infer that $|z_1(t)| < k_{b_1}$, $\forall t > 0$. Since $y(t) = z_1(t) + y_r(t)$, and $|y_r(t)| \le A_0$ in Assumption 3, we obtain that $|y(t)| \le |z_1(t)| + |y_r(t)| < k_{b_1} + A_0 = k_{c_1}$. Therefore, we can conclude that the output constraint is never violated.

(ii) From $\dot{V}_\rho \le 0$ and Lemma 1, we infer that $\bar{z}_n$, $\hat{\Theta}$, $\hat{\varrho}$, ε are bounded. Since z_1 and y_r are bounded, y is also bounded. Then, from (4) and (5), we conclude that ξ and Ξ are bounded as A_0 is Hurwitz. Assumption 2 and (6) imply that $\bar{\lambda}_{m+1}$ are bounded. From the coordinate change (15), it follows that

$$v_{m,i} = z_i + \hat{\varrho}y_r^{(i-1)} + \alpha_{i-1}(y, \xi, \Xi, \hat{\Theta}, \hat{\varrho}, \bar{\lambda}_{m+i-1}, \bar{y}_r^{(i-2)}), \quad i = 2, 3, ..., \rho \quad (49)$$

For $i = 2$, the boundedness of $\bar{\lambda}_{m+1}$, along with the boundedness of z_2 and $y, \xi, \Xi, \hat{\Theta}, \hat{\varrho}, y_r, \dot{y}_r$, proves that $v_{m,2}$ is bounded. From (7), it follows that λ_{m+2} is bounded. Following the same procedure recursively, the boundedness of λ is established. Finally, from (8) and the boundedness of $\xi, \Xi, \lambda, \varepsilon$, we conclude that x is bounded. Furthermore, $u(t)$ is bounded. Hence, all closed loop signals are bounded.

(iii) By applying the LaSalle-Yoshizawa theorem to (48), it follows that $z_i \to 0$ as $t \to \infty$ for $i = 1, ..., n$, which implies that $y(t) \to y_d(t)$ as $t \to \infty$. ∎

Remark 1: From Theorem 1, we know that the sufficient condition to avoid output constraint transgression for the plant (1) using BLFs is

$$\bar{z}_n(0) \in \Omega_{z_0} := \{\bar{z}_n \in R^n : |z_1| < k_{b_1}\} \quad (50)$$

which implies that only the initial value of z_1 is needed to be constrained. Compared with this, if we consider the case using QLFs for the plant (1) as in [11], we have the follow conclusion

$$|z_i(t)| \le \sqrt{V(0)} \quad (51)$$

where

$$V(0) = \sum_{i=1}^{\rho}\frac{1}{2}z_i^2(0) + \frac{1}{2}\tilde{\Theta}^T(0)\Gamma^{-1}\tilde{\Theta}(0) + \frac{|b_m|}{2\gamma_\varrho}\tilde{\varrho}^2(0) + \sum_{i=1}^{\rho}\frac{1}{2d_i}\varepsilon^T(0)P\varepsilon(0) \quad (52)$$

From (51) and (52), it can be seen that the initial conditions (50) is not sufficient to guarantee that $|z_1(t)| < k_{b_1}$ for the case using QLFs. Besides the constraint on $z_1(0)$, there should be other constraints on $z_i(0)$, $i = 2, ..., n$, $\tilde{\Theta}(0)$, $\tilde{\varrho}(0)$, and $\varepsilon(0)$ as well, which is more restrictive than (50).

V. Simulation Results

In this section, the feasibility and effectiveness of the proposed approach are illustrated by an example. Consider a second-order output feedback system as follows

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \theta x_1 + u\\ y &= x_1\end{aligned} \quad (53)$$

where $\theta = 1.0$. The objective is for y to track the desired trajectory $y_r = 0.5\sin(0.5t)$, subject to the output constraint $|y| < k_{c_1} = 0.8$. Therefore, $k_{b_1} = 0.8 - 0.5 = 0.3$.

The initial conditions and the control design parameters are chosen as: $x_1(0) = 0.2$, $x_2(0) = 1.5$, $\xi_i(0) = \xi_i(0) =$

$v_{0,i}(0) = \hat{\theta}(0) = 0.0$, $i = 1, 2$, $k_1 = k_2 = 1.0$, $c_1 = c_2 = 2.0$, $d_1 = d_2 = 0.5$, $\Gamma = 1.0$.

The simulation results are shown in Figs. 2-4. From Fig.2, it can be seen that the output y remains within the constraint $|y| < k_{c_1} = 0.8$ and tracks the desired trajectory y_r asymptotically when the BLF is used. However, when the QLF is used under the same initial conditions, the output constrain is violated. The tracking error $z_1 = y - y_r$ and the control u using BLF are shown in Fig. 3. In addition, the boundedness of other signals x_2 and $\hat{\theta}$ can be seen in Fig. 4.

VI. Conclusion

In this paper, the tracking problem for a class of parametric output feedback nonlinear systems with output constraint has been addressed by the combination of adaptive observer backstepping method and Barrier Lyapunov Function (BLF). Compared with Quadratic Lyapunov Function (QLFs), We have shown that QLFs result in more conservative initial conditions than those resulting from BLFs. The stability analysis for the closed-loop system has been provided and the feasibility of the proposed approach was illustrated by a numerical example.

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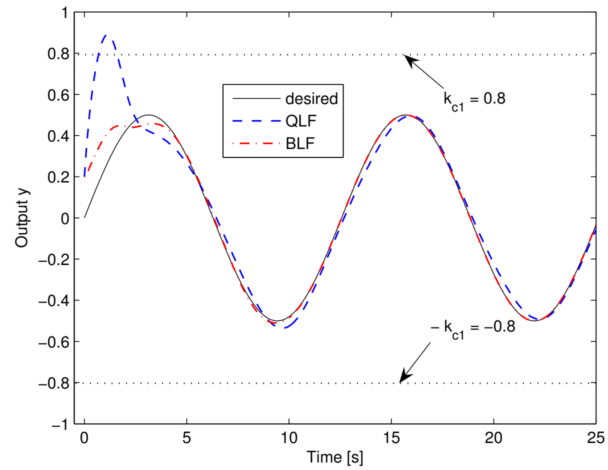

Fig. 2. Comparisons of output tracking using QLF and BLF.

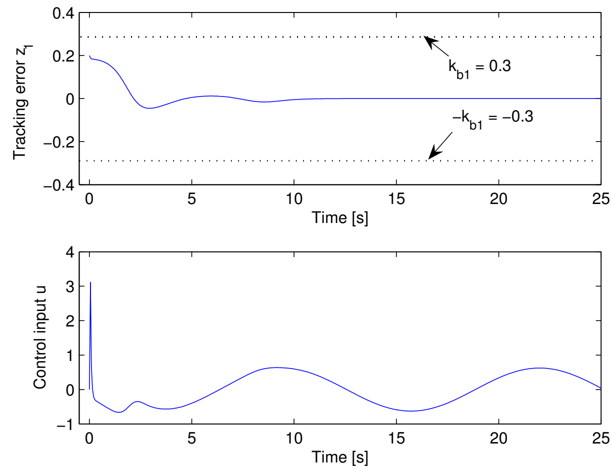

Fig. 3. Tracking error z_1 and control input u using BLF.

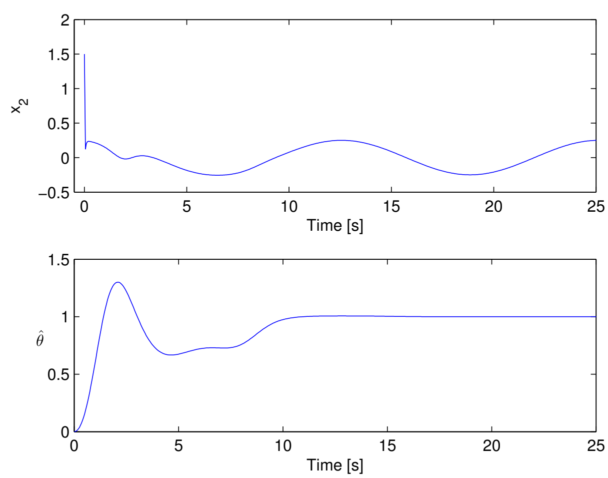

Fig. 4. State x_2 and parameter estimate $\hat{\theta}$ using BLF.