Brief Papers

Adaptive Neural Control for Output Feedback Nonlinear Systems Using a Barrier Lyapunov Function

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***Abstract*—In this brief, adaptive neural control is presented for a class of output feedback nonlinear systems in the presence of unknown functions. The unknown functions are handled via on-line neural network (NN) control using only output measurements. A barrier Lyapunov function (BLF) is introduced to address two open and challenging problems in the neuro-control area: 1) for any initial compact set, how to determine *a priori* the compact superset, on which NN approximation is valid; and 2) how to ensure that the arguments of the unknown functions remain within the specified compact superset. By ensuring boundedness of the BLF, we actively constrain the argument of the unknown functions to remain within a compact superset such that the NN approximation conditions hold. The semiglobal boundedness of all closed-loop signals is ensured, and the tracking error converges to a neighborhood of zero. Simulation results demonstrate the effectiveness of the proposed approach.**

***Index Terms*—Barrier function, neural networks (NNs), output feedback nonlinear systems, unknown functions.**

I. Introduction

Since the seminal work [15], great progress has been witnessed in neural network (NN) control of nonlinear systems, which has evolved to become a well-established technique in advanced adaptive control. Adaptive NN control approaches based on Lyapunov's stability theory has been investigated for nonlinear systems with matching [1], [4], [14], [21], and nonmatching conditions [18], [19], as well as systems with output feedback requirement [2], [10], [11], [22], [23]. The main trend in recent neural control research is to integrate NN, including multilayer networks [14], radial basis function networks [21], and recurrent ones [20], with main nonlinear control design methodologies. Such integration significantly enhances the capability of control methods in handling many practical systems that are characterized by nonlinearity, uncertainty, and complexity [5], [7], [13].

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It is well known that NN approximation-based control relies on universal approximation property in a compact set in order to approximate unknown nonlinearities in the plant dynamics. For any initial compact set Ω^0, as long as the arguments of the unknown function start from Ω^0 and remain within a compact superset Ω, as shown in Fig. 1 [8], NN approximation is valid.

Therefore, how to determine *a priori* the compact superset Ω and how to ensure the arguments of the unknown function remain within the compact superset Ω, are two open and challenging problems in the neuro-control area [2]. One method of ensuring that the NN approximation condition holds is by careful selection of the control parameters, via rigorous transient performance analysis, so that the system states do not transgress the compact superset of approximation Ω [6], [8], but the compact superset Ω is only given qualitatively, not quantitively. Another method is to rely on sliding mode control operating in parallel to the approximation-based control, such that the compact superset Ω is rendered positively invariant [5], [27]. The compact superset Ω can be specified *a priori*, but there exist some implementation issues, such as the fixed-point problem in the input signal.

Recently, the design of barrier functions in Lyapunov synthesis has been proposed for constraint handling in Brunovsky-type systems [16], nonlinear systems in strict feedback form [25], and electrostatic microactuators [24]. Unlike conventional Lyapunov functions, which are well-defined over the entire domain and radially unbounded for global stability, a barrier Lyapunov function (BLF) possesses the special property of approaching infinity whenever its arguments approach some limits. By ensuring boundedness of the BLF along the system trajectories, transgression of constraints is prevented. We note that the BLF-based control design methodology appears very promising in providing yet another means of tackling the NN approximation-based control problems, by actively constraining the states of the system to remain within the compact set of approximation.

In this brief, we present adaptive neural control for a class of output feedback nonlinear systems subject to function uncertainties. The unknown functions are compensated for via online NN function approximation using only output measurements. To address two important neural control concerns mentioned above, the BLF is incorporated into Lyapunov synthesis by following the constructive procedures of adaptive observer backstepping design [12]. First, for any initial compact set Ω^0 where the argument of the unknown function belongs to, we can always construct an *a priori* compact superset Ω. Second, by ensuring the boundedness of the BLF, we guarantee that the argument of the unknown function remains within the compact superset Ω, on which the NN approximation is valid. Then, the stable output tracking with guaranteed performance bounds can be achieved in the semi-global sense.

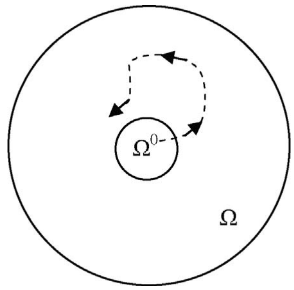

Fig. 1. Compact sets for NN approximation [8].

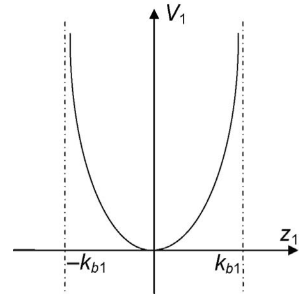

Fig. 2. Schematic illustration of barrier functions.

II. Problem Formulation and Preliminaries

A. Problem Formulation

Consider a class of output feedback nonlinear systems described by

$$\begin{aligned} \dot{x}_1 &= x_2 + f_1^0(y) + f_1(y) + d_1(t) \\ &\vdots \\ \dot{x}_{\rho-1} &= x_\rho + f_{\rho-1}^0(y) + f_{\rho-1}(y) + d_{\rho-1}(t) \\ \dot{x}_\rho &= x_{\rho+1} + f_\rho^0(y) + f_\rho(y) + d_\rho(t) + b_m u \\ &\vdots \\ \dot{x}_{n-1} &= x_n + f_{n-1}^0(y) + f_{n-1}(y) + d_{n-1}(t) + b_1 u \\ \dot{x}_n &= f_n^0(y) + f_n(y) + d_n(t) + b_0 u \\ y &= x_1 \end{aligned} \tag{1}$$

where $x_1, \ldots, x_n$ are system states, y and u are the output and input, respectively; $f_i^0(y)$, $i = 1, \ldots, n$ are known smooth functions, which represent nominal parts of the plant and may be available using some prior physical or expert information ($f_i^0(y) = 0$ if no prior knowledge of the nonlinearity); $f_i(y)$, $i = 1, \ldots, n$ are unknown smooth functions, which represent model uncertainties due to modeling errors or unmodeled dynamics; $d_i(t)$ are bounded time-varying disturbances with unknown constant bounds; $b_m, \ldots, b_0$ are uncertain constant parameters.

Remark 1: Several cases when $f_i(y)$ in (1) satisfy the linear-in-the-parameters (LIP) condition have been intensively investigated in [3], [12], [26]. When uncertain $f_i(y)$ do not satisfy LIP condition, adaptive observer backstepping control using NNs has been presented in [2], without addressing the two open and challenging problems in the neuro-control area mentioned in Section I.

Assumption 1: The unknown disturbance $d_i(t)$ satisfies $|d_i(t)| \le \bar{d}_i$, where $\bar{d}_i$ is an unknown constant.

Assumption 2: The sign of b_m is known.

Assumption 3: The relative degree $\rho = n - m$ is known and the system is minimum phase, i.e., the polynomial $B(s) = b_m s^m + \cdots + b_1 s + b_0$ is Hurwitz.

Assumption 4: There exist positive constants $Y_0, \underline{Y}_0, \bar{Y}_0, Y_1, Y_2, \ldots, Y_\rho$ satisfying $\max\{\underline{Y}_0, \bar{Y}_0\} \le Y_0$ such that the reference signal $y_r(t)$ and its ρth order derivatives are known and bounded, which satisfy $-\underline{Y}_0 \le y_r(t) \le \bar{Y}_0$, $|\dot{y}_r(t)| < Y_1$, $|\ddot{y}_r(t)| < Y_2, \ldots, |y_r^{(\rho)}(t)| < Y_\rho$, $\forall t \ge 0$.

Assuming that only the output signal y is measured, the control objective is to drive the output y to track the given reference signal $y_r(t)$ within a neighborhood of zero, while keeping all of the signals in the closed-loop system bounded.

B. Function Approximation

In this brief, the following radial basis function NNs [9], [17] is used to approximate the continuous function $f_i(y) : \mathbb{R} \to \mathbb{R}$:

$$f_i^{nn}(y, \theta_i) = \phi_i^T(y)\theta_i \tag{2}$$

where the input $y \in \Omega_y \subset \mathbb{R}$; the weight vector $\theta_i = [\theta_{i1}, \ldots, \theta_{il_i}]^T$ with the NN node number l_i; the vector of smooth basis functions $\phi_i = [\phi_{i1}, \phi_{i2}, \ldots, \phi_{il_i}]^T \in R^{l_i}$, $\phi_{ij}(y)$ being chosen as the commonly used Gaussian functions $\phi_{ij}(y) = \exp\left[\frac{-(y-\mu_{ij})^T(y-\mu_{ij})}{\eta_i^2}\right]$, $j = 1, 2, \ldots, l_i$, where μ_{ij} is the center of the receptive field and η_i is the width of the Gaussian function.

It has been proven in [21] that network (2) can approximate any smooth function over a compact set $\Omega_y \subset \mathbb{R}$ to arbitrarily any accuracy as

$$f_i(y, \theta_i) = \phi_i^T(y)\theta_i^* + \varepsilon_i(y) \tag{3}$$

where θ_i^* are ideal constant weights, and the approximation error $\varepsilon_i(y)$ satisfies $|\varepsilon_i(y)| \le \varepsilon_i^*$ with constant $\varepsilon_i^* > 0$ for all $y \in \Omega_y$.

The ideal weight vector θ_i^*, an "artificial" quantity required for analytical purposes, is defined as the value of θ_i that minimizes $|\varepsilon_i(y)|$, $\forall y \in \Omega_y$, that is

$$\theta_i^* = \arg\min_{(\theta_i)}\left[\sup_{y\in\Omega_y} |\phi_i^T(y)\theta_i - f(y)|\right]. \tag{4}$$

C. Barrier Lyapunov Function

Definition 1: [25] A BLF is a scalar function $V(x)$, defined with respect to the system $\dot{x} = f(x)$ on an open region $\mathcal{D}$ containing the origin, that is continuous, positive definite, has continuous first-order partial derivatives at every point of $\mathcal{D}$, has the property $V(x) \to \infty$ as x approaches the boundary of $\mathcal{D}$, and satisfies $V(x(t)) \le b$ $\forall t \ge 0$ along the solution of $\dot{x} = f(x)$ for $x(0) \in \mathcal{D}$ and some positive constant b.

In this brief, the following BLF candidate considered in [16], [25] is used throughout this brief:

$$V_1 = \frac{1}{2}\log\frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2} \tag{5}$$

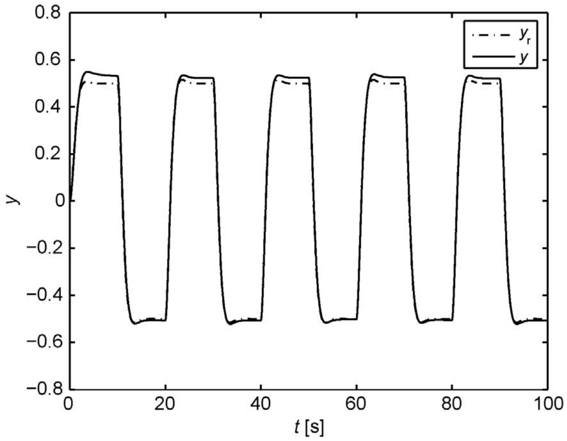

Fig. 3. Output tracking performance.

where $\log(\cdot)$ denotes the natural logarithm of $\cdot$, and k_{b_1} the constraint on z_1, i.e., $|z_1| < k_{b_1}$. As seen from the schematic illustration of $V_1(z_1)$ in Fig. 2, the BLF escapes to infinity at $|z_1| = k_{b_1}$. It can be shown that V_1 is positive definite and C^1 continuous in the set $|z_1| < k_{b_1}$, and thus, a valid Lyapunov function candidate in the set $|z_1| < k_{b_1}$.

Lemma 1: For any positive constant k_{b_1}, let $\mathcal{Z}_1 := \{z_1 \in \mathbb{R} : |z_1| < k_{b_1}\} \subset \mathbb{R}$ and $\mathcal{N} := \mathbb{R}^l \times \mathcal{Z}_1 \subset \mathbb{R}^{l+1}$ be open sets. Consider the system

$$\dot{\eta} = h(t, \eta) \tag{6}$$

where $\eta := [w, z_1]^T \in \mathcal{N}$ is the state, and the function $h : \mathbb{R}_+ \times \mathcal{N} \to \mathbb{R}^{l+1}$ is piecewise continuous in t and locally Lipschitz in z_1, uniformly in t, on $\mathbb{R}_+ \times \mathcal{N}$. Suppose that there exist continuously differentiable and positive definite functions $U : \mathbb{R}^l \to \mathbb{R}_+$ and $V_1 : \mathcal{Z}_1 \to \mathcal{R}_+$, $i = 1, \ldots, n$, such that

$$V_1(z_1) \to \infty \quad \text{as} \quad |z_1| \to k_{b_1} \tag{7}$$

$$\gamma_1(\|w\|) \le U(w) \le \gamma_2(\|w\|) \tag{8}$$

with γ_1 and γ_2 as class K_∞ functions. Let $V(\eta) := V_1(z_1) + U(w)$, and $z_1(0) \in \mathcal{Z}_1$. If the inequality holds

$$\dot{V} = \frac{\partial V}{\partial \eta} h \le -\mu V + \lambda \tag{9}$$

in the set $\eta \in \mathcal{N}$ and μ, λ are positive constants, then w remains bounded and $z_1(t) \in \mathcal{Z}_1$, $\forall t \in [0, \infty)$.

Proof: The proof is omitted here due to the limited space. Interested readers can follow the similar procedures of the proof of Lemma 1 in [25]. ■

Lemma 2: For any positive constant k_{b_1}, the following inequality holds for all z_1 in the interval $|z_1| < k_{b_1}$:

$$\log \frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2} < \frac{z_1^2}{k_{b_1}^2 - z_1^2}. \tag{10}$$

Proof: The proof is omitted here due to the limited space. ■

III. State Estimation Filter and Observer Design

Since only the output signal y is measured, some filters should be designed first, which will provide "virtual estimates" of the unmeasured state variables $x_2, \ldots, x_n$. Substituting (3) into (1) and after some manipulations, we obtain that

$$\dot{x} = Ax + F^0(y) + \Phi(y)\theta^* + \Delta(y, t) + \begin{bmatrix} 0 \\ b \end{bmatrix} u \tag{11}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

$$F^0(y) = \begin{bmatrix} f_1^0(y) \\ \vdots \\ f_n^0(y) \end{bmatrix} \in \mathbb{R}^{n\times 1}, \quad \theta^* = \begin{bmatrix} \theta_1^* \\ \vdots \\ \theta_n^* \end{bmatrix} \in \mathbb{R}^{ln\times 1}$$

$$\Phi(y) = \begin{bmatrix} \Phi_1^T(y) \\ \vdots \\ \Phi_n^T(y) \end{bmatrix} = \begin{bmatrix} \phi_1^T(y) & 0 & \cdots & 0 \\ 0 & \phi_2^T(y) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_n^T(y) \end{bmatrix} \in \mathbb{R}^{n\times ln}$$

$$\Delta(y, t) = \begin{bmatrix} \Delta_1(y, t) \\ \vdots \\ \Delta_n(y, t) \end{bmatrix} = \begin{bmatrix} \varepsilon_1(y) + d_1(t) \\ \vdots \\ \varepsilon_n(y) + d_n(t) \end{bmatrix} \in \mathbb{R}^{n\times 1}$$

$$b = \begin{bmatrix} b_m \\ \vdots \\ b_0 \end{bmatrix} \in \mathbb{R}^{(m+1)\times 1}. \tag{12}$$

From Assumption 1 and (3), we know that $|\Delta_i(y, t)| \le \varepsilon_i^* + \bar{d}_i < \psi$, where ψ is an unknown bounding parameter and will be estimated by $\hat{\psi}$.

Choose the K-filters [12] as follows:

$$\dot{\xi} = A_0\xi + ky + F^0(y) \tag{13}$$

$$\dot{\Xi} = A_0\Xi + \Phi(y) \tag{14}$$

$$\dot{\lambda} = A_0\lambda + e_n u \tag{15}$$

$$v_i = A_0^i \lambda, \quad i = 0, 1, \ldots, m \tag{16}$$

where $k = [k_1, \ldots, k_n]^T$ such that $A_0 = A - ke_1^T$ is Hurwitz, A_0^i denotes the ith power of the matrix A_0, and e_i is the ith coordinate vector in $\mathbb{R}^n$.

By constructing the state estimates as follows:

$$\hat{x}(t) = \xi + \Xi\theta^* + \sum_0^m b_i v_i. \tag{17}$$

It is straightforward to verify that the dynamics of the observation error, $\tilde{x} = x - \hat{x}$, are given by

$$\dot{\tilde{x}} = A_0\tilde{x} + \Delta(y, t). \tag{18}$$

Since A_0 is Hurwitz, it can be shown that the error system (18) with state $\tilde{x}$ is input state stable with respect to the term

$\Delta(y,t)$. Furthermore, system (1) can be represented as

$$\dot{y} = b_m v_{m,2} + \xi_2 + f_1^0(y) + \bar{\Omega}^T\Theta + \Delta_1(y,t) + \tilde{x}_2 \tag{19}$$

$$\dot{v}_{m,i} = v_{m,i+1} - k_i v_{m,1}, \quad i = 2, 3, \ldots, \rho - 1 \tag{20}$$

$$\dot{v}_{m,\rho} = v_{m,\rho+1} - k_\rho v_{m,1} + u \tag{21}$$

with $\Theta = [b_m, \ldots, b_0, \theta^{*T}]^T$, $\Omega = [v_{m,2}, v_{m-1,2}, \ldots, v_{0,2}, \Xi_2 + \Phi_1^T]^T$, and $\bar{\Omega} = [0, v_{m-1,2}, \ldots, v_{0,2}, \Xi_2 + \Phi_1^T]^T$, where $\tilde{x}_2$, $v_{i,2}$, ξ_2, and Ξ_2 denote the second entries of $\tilde{x}$, v_i, ξ, and Ξ, respectively, and y, v_i, ξ, and Ξ are all available signals.

IV. Adaptive Observer Backstepping Design

In this section, we present the adaptive control design using the backstepping technique. Since adaptive backstepping design is mature, we omit the details. Interested readers are referred to [12]. Define the following error coordinates: $z_1 = y - y_r$ and $z_i = v_{m,i} - \alpha_{i-1} - \hat{\varrho} y_r^{(i-1)}$, $i = 2, 3, \ldots, \rho$, where $\hat{\varrho}$ is an estimate of $\varrho = \frac{1}{b_m}$ and α_{i-1} is the stabilizing functions to be designed.

For any initial compact set $\Omega_y^0 := \{y \in \mathbb{R} \big| |y| \le k_0, k_0 > 0\} \subset \mathbb{R}$, which $y(0)$ belongs to, we can always specify another compact set $\Omega_y := \{y \in \mathbb{R} \big| |y| \le k_{c_1}, k_{c_1} > k_0 + Y_0 + |y_r(0)|\} \subset \mathbb{R}$, which is a superset of Ω_y^0 and can be made as large as desired. As long as the input variable of the NNs, y, remains within this prefixed compact Ω_y, the NN approximation is valid. Borrowing the idea of the BLF-based control in [24], [25], to design a control that does not drive y out of the interval $|y| < k_{c_1}$, we require that $|z_1| < k_{b_1}$ with $k_{b_1} = k_{c_1} - Y_0$ and choose the following Lyapunov function candidates:

$$V_1 = \frac{1}{2}\log\frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2} + \frac{1}{2}\tilde{\Theta}^T\Gamma^{-1}\tilde{\Theta} + \frac{|b_m|}{2\gamma_\varrho}\tilde{\varrho}^2 + \frac{1}{2\gamma_\psi}\tilde{\psi}^2 + \frac{1}{2\gamma_1}\tilde{x}^T P\tilde{x} \tag{22}$$

$$V_i = V_{i-1} + \frac{1}{2}z_i^2 + \frac{1}{2\gamma_i}\tilde{x}^T P\tilde{x}, \quad i = 2, \ldots, \rho \tag{23}$$

where $\tilde{\Theta} = \Theta - \hat{\Theta}$, $\hat{\Theta}$ is the estimate of Θ, $\tilde{\psi} = \psi - \hat{\psi}$, Γ is a positive definite design matrix, γ_ϱ, γ_ψ, and γ_i are positive design parameters, and P is a definite positive matrix such that $PA_0 + A_0^T P = -I$, $P = P^T > 0$. The adaptive backstepping control is designed as follows:

$$\alpha_1 = \hat{\varrho}\left[-c_1 z_1 - \xi_2 - f_1^0(y) - \bar{\Omega}^T\hat{\Theta} - \frac{\gamma_1 z_1}{k_{b_1}^2 - z_1^2} - \hat{\psi}\tanh\left(\frac{\frac{z_1}{k_{b_1}^2 - z_1^2}}{\delta_1}\right)\right] \tag{24}$$

$$\alpha_2 = -\frac{\hat{b}_m z_1}{k_{b_1}^2 - z_1^2} - c_2 z_2 + \beta_2 + \frac{\partial\alpha_1}{\partial\hat{\Theta}}\Gamma\tau_{2\theta} + \frac{\partial\alpha_1}{\partial\hat{\psi}}\gamma_\psi\tau_{2\psi} - \hat{\psi}\frac{\partial\alpha_1}{\partial y}\tanh\left(\frac{z_2\frac{\partial\alpha_1}{\partial y}}{\delta_2}\right) - \gamma_2\left(\frac{\partial\alpha_1}{\partial y}\right)^2 z_2 \tag{25}$$

$$\begin{aligned}\alpha_i = & -z_{i-1} - c_i z_i + \beta_i + \frac{\partial\alpha_{i-1}}{\partial\hat{\Theta}}\Gamma\tau_{i\theta} + \frac{\partial\alpha_{i-1}}{\partial\hat{\psi}}\gamma_\psi\tau_{i\psi} \\ & - \hat{\psi}\frac{\partial\alpha_{i-1}}{\partial y}\tanh\left(\frac{z_i\frac{\partial\alpha_{i-1}}{\partial y}}{\delta_i}\right) - \gamma_i\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2 z_i \\ & - \left(\sum_{k=2}^{i-1} z_k\frac{\partial\alpha_{k-1}}{\partial\hat{\Theta}}\right)\Gamma\frac{\partial\alpha_{i-1}}{\partial y}\Omega \\ & - \left(\sum_{k=2}^{i-1} z_k\frac{\partial\alpha_{k-1}}{\partial\hat{\psi}}\right)\gamma_\psi\frac{\partial\alpha_{i-1}}{\partial y}\tanh\left(\frac{z_i\frac{\partial\alpha_{i-1}}{\partial y}}{\delta_i}\right) \\ & i = 3, \ldots, \rho\end{aligned} \tag{26}$$

$$\begin{aligned}\beta_i = & \frac{\partial\alpha_{i-1}}{\partial y}\left(\xi_2 + f_1^0(y) + \Omega^T\hat{\Theta}\right) + k_i v_{m,1} \\ & + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_r^{(j-1)}}y_r^{(j)} + (\frac{\partial\alpha_{i-1}}{\partial\hat{\varrho}} + y_r^{(i-1)})\dot{\hat{\varrho}} \\ & + \sum_{j=1}^{m+i-1}\frac{\partial\alpha_{i-1}}{\partial\lambda_j}(-k_j\lambda_1 + \lambda_{j+1}) + \frac{\partial\alpha_{i-1}}{\partial\xi}(A_0\xi \\ & + ky + \Psi(y)) + \frac{\partial\alpha_{i-1}}{\partial\Xi}(A_0\Xi^T + \Phi(y)) \\ & i = 2, \ldots \rho\end{aligned} \tag{27}$$

$$\dot{\hat{\varrho}} = -\gamma_\varrho\left[\mathrm{sign}(b_m)(\dot{y}_r + \bar{\alpha}_1)\frac{z_1}{k_{b_1}^2 - z_1^2} + \sigma_\varrho\hat{\varrho}\right] \tag{28}$$

$$\tau_{1\theta} = \frac{z_1}{k_{b_1}^2 - z_1^2}[\Omega - \hat{\varrho}(\dot{y}_r + \bar{\alpha}_1)e_1] - \sigma_\theta\hat{\Theta} \tag{29}$$

$$\tau_{1\psi} = \frac{z_1}{k_{b_1}^2 - z_1^2}\tanh\left(\frac{\frac{z_1}{k_{b_1}^2 - z_1^2}}{\delta_1}\right) - \sigma_\psi\hat{\psi} \tag{30}$$

$$\tau_{i\theta} = \tau_{(i-1)\theta} - z_i\frac{\partial\alpha_{i-1}}{\partial y}\Omega, \quad i = 2, \ldots, \rho \tag{31}$$

$$\tau_{i\psi} = \tau_{(i-1)\psi} + z_i\frac{\partial\alpha_{i-1}}{\partial y}\tanh\left(\frac{z_i\frac{\partial\alpha_{i-1}}{\partial y}}{\delta_i}\right), \quad i = 2, \ldots, \rho \tag{32}$$

$$u = \alpha_\rho - v_{m,\rho+1} + \hat{\varrho}y_r^{(\rho)} \tag{33}$$

$$\dot{\hat{\Theta}} = \Gamma\tau_{\rho\theta} \tag{34}$$

$$\dot{\hat{\psi}} = \gamma_\psi\tau_{\rho\psi} \tag{35}$$

where c_i and δ_i are positive design parameters.

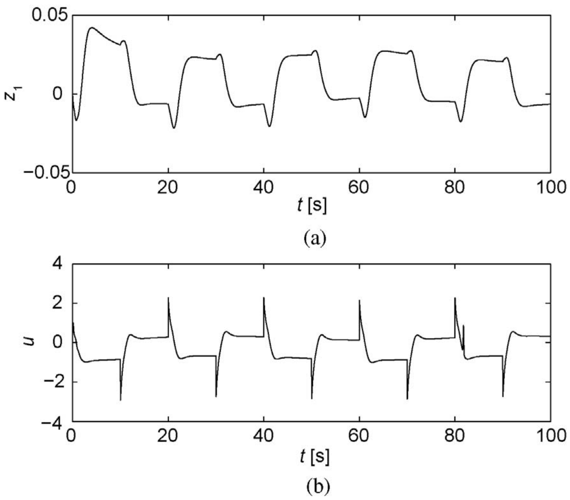

Fig. 4. (a) Tracking error z_1. (b) Control input u.

Then, the derivative of V_ρ is given by

$$\begin{aligned}\dot{V}_\rho \ \le\ & -\frac{c_1 z_1^2}{k_{b_1}^2 - z_1^2} - \sum_{i=2}^{\rho} c_i z_i^2 - \frac{\sigma_\theta}{2}\|\tilde{\Theta}\|^2 - \frac{\sigma_\psi}{2}\tilde{\psi}^2 \\ & -\frac{\sigma_\varrho}{2}|b_m|\tilde{\varrho}^2 - \sum_{i=1}^{\rho}\frac{1}{4\gamma_i}\tilde{x}^T\tilde{x} + \frac{\sigma_\theta}{2}\|\Theta\|^2 + \frac{\sigma_\psi}{2}\psi^2 \\ & +\frac{\sigma_\varrho}{2}|b_m|\varrho^2 + \sum_{i=1}^{\rho} 0.2785\delta_i\psi. \qquad (36)\end{aligned}$$

Theorem 1: Consider the closed-loop system consisting of the plant (1), filters (13)–(16), stabilizing functions (24)–(26), control law (33), and adaptation laws (28) and (34), under Assumptions 1–4. Then, for any initial compact set Ω_y^0, which $y(0)$ belongs to:

1) there always exists a sufficiently large compact set Ω_y, such that $y(t) \in \Omega_y$, $\forall t > 0$;
2) all closed loop signals are bounded;
3) the output tracking error converges to a neighborhood of zero, which can be made arbitrarily small by appropriate selection of design parameters.

Proof:

1) According to Lemma 2, $-\frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2} < -\log\frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2}$ in the set $|z_1| < k_{b_1}$. Therefore, (36) can be further represented as

$$\begin{aligned}\dot{V}_\rho \ \le\ & -c_1\log\frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2} - \sum_{i=2}^{\rho} c_i z_i^2 - \frac{\sigma_\theta}{2}\|\tilde{\Theta}\|^2 \\ & -\frac{\sigma_\psi}{2}\tilde{\psi}^2 - \frac{\sigma_\varrho}{2}|b_m|\tilde{\varrho}^2 - \sum_{i=1}^{\rho}\frac{1}{4\gamma_i}\tilde{x}^T\tilde{x} + \frac{\sigma_\theta}{2}\|\Theta\|^2 \\ & +\frac{\sigma_\psi}{2}\psi^2 + \frac{\sigma_\varrho}{2}|b_m|\varrho^2 + \sum_{i=1}^{\rho} 0.2785\delta_i\psi \\ \le\ & -\mu_1 V_\rho + \mu_2 \qquad (37)\end{aligned}$$

in the set $|z_1| < k_{b_1}$ with $\mu_1 = \min\left\{2c_i, \frac{\sigma_\theta}{\lambda_{\max}(\Gamma^{-1})}, \sigma_\varrho\gamma_\varrho, \sigma_\psi\gamma_\psi, \frac{1}{2\lambda_{\max}(P)}\right\}$ and $\mu_2 = \frac{\sigma_\theta}{2}\|\Theta\|^2 + \frac{\sigma_\psi}{2}\psi^2 + \frac{\sigma_\varrho}{2}|b_m|\varrho^2 + \sum_{i=1}^{\rho} 0.2785\delta_i\psi$.

We can rewrite the closed loop system consisting of the plant (1), filters (13)–(16), stabilizing functions (24)–(26), control law (33) and adaptation laws (28), (34), as $\dot{\eta} = h(t,\eta)$, where $\eta = [\bar{z}_n^T, \tilde{\Theta}^T, \tilde{\varrho}, \tilde{\psi}, \tilde{x}^T]^T$. Then, it can be shown that $h(t,\eta)$ satisfies the conditions in Lemma 1 for $\eta \in \Omega = \left\{\bar{z}_n \in \mathbb{R}^n, \tilde{\Theta} \in \mathbb{R}^{ln+m+1}, \tilde{\varrho} \in \mathbb{R}, \tilde{\psi} \in \mathbb{R}, \tilde{x} \in \mathbb{R}^n \,\middle|\, |z_1| < k_{b_1}\right\}$. Since $z_1(0) = y(0) - y_r(0)$, $y(0) \le k_0$ in the definition of Ω_y^0 and $k_{c_1} > k_0 + Y_0 + |y_r(0)|$ in the definition of Ω_y, we obtain that $|z_1(0)| < k_{b_1}$. Therefore, we can conclude that the set Ω is an invariant set. Together with (37), we infer, from Lemma 1, that $|z_1(t)| < k_{b_1}$, $\forall t > 0$. Since $y(t) = z_1(t) + y_r(t)$ and $|y_r(t)| \le Y_0$ in Assumption 4, we obtain that $|y(t)| \le |z_1(t)| + |y_r(t)| < k_{b_1} + Y_0 = k_{c_1}$, $\forall t > 0$. As such, we can conclude that for any initial compact set Ω_y^0, which $y(0)$ belongs to, there always exists a sufficiently large compact set Ω_y, such that $y \in \Omega_y$, $\forall t > 0$.

2) Let $\mu_0 = \frac{\mu_2}{\mu_1}$, then (37) satisfies

$$0 \le V_\rho(t) \le \mu_0 + (V_\rho(0) - \mu_0)e^{-\mu_1 t} \le \mu_0 + V_\rho(0). \qquad (38)$$

Therefore, from (23), we infer that $\bar{z}_n$, $\hat{\Theta}$, $\hat{\varrho}$, $\hat{\psi}$, $\tilde{x}$ are bounded. Since z_1 and y_r are bounded, y is also bounded. Then, from (13) and (14), we conclude that ξ and Ξ are bounded as A_0 is Hurwitz. Assumption 3 and (15) imply that $\bar{\lambda}_{m+1}$ are bounded. It follows that:

$$\begin{aligned}v_{m,i} \ =\ & z_i + \hat{\varrho}y_r^{(i-1)} + \alpha_{i-1}(y, \xi, \Xi, \hat{\Theta}, \hat{\varrho}, \hat{\psi}, \\ & \bar{\lambda}_{m+i-1}, \bar{y}_r^{(i-2)}), \quad i = 2, 3, \ldots, \rho. \qquad (39)\end{aligned}$$

For $i = 2$, the boundedness of $\bar{\lambda}_{m+1}$, along with the boundedness of z_2 and $y, \xi, \Xi, \hat{\Theta}, \hat{\varrho}, \hat{\psi}, y_r, \dot{y}_r$, proves that $v_{m,2}$ is bounded. From (16), it follows that λ_{m+2} is bounded. Following the same procedure recursively, the boundedness of λ is established. Finally, from (17) and the boundedness of $\xi, \Xi, \lambda, \tilde{x}$, we conclude that x is bounded. Furthermore, $u(t)$ is bounded. Hence, all closed loop signals are bounded.

3) From (23) and (38), we obtain that

$$\frac{1}{2}\log\frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2} \le \mu_0 + (V_\rho(0) - \mu_0)e^{-\mu_1 t}. \qquad (40)$$

Taking exponentials on both sides of (40) results in

$$\frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2} \le e^{2[\mu_0 + (V_\rho(0) - \mu_0)e^{-\mu_1 t}]}. \qquad (41)$$

Since $|z_1(t)| < k_{b_1}$ is obtained in (i), we have, that $k_{b_1}^2 - z_1^2 > 0$. Multiplying both sides by $(k_{b_1}^2 - z_1^2)$ and after some manipulations lead to

$$|z_1(t)| \le k_{b_1}\sqrt{1 - e^{-2[\mu_0 + (V_\rho(0) - \mu_0)e^{-\mu_1 t}]}}. \qquad (42)$$

It follows that given any $\mu > k_{b_1}\sqrt{1 - e^{-2\mu_0}}$, there exists T such that for all $t > T$, $|z_1(t)| \le \mu$. As $t \to \infty$, $|z_1(t)| \le k_{b_1}\sqrt{1 - e^{-2\mu_0}}$, which implies that

$$|y - y_r| \le k_{b_1}\sqrt{1 - e^{-2\mu_0}}, \quad \text{as} \quad t \to \infty. \qquad (43)$$

Due to $\mu_0 = \frac{\mu_2}{\mu_1}$, and from the definitions of μ_1 and μ_2 (37), we see that $y - y_r$ can be made arbitrarily small by appropriate selection of design parameters. ■

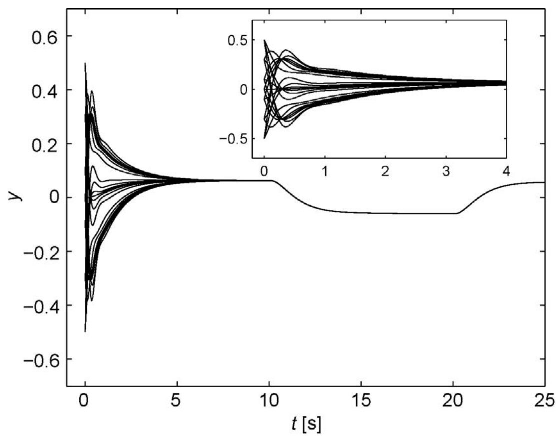

Fig. 5. Output trajectories for different initial conditions.

V. Simulation Results

Consider a second-order output feedback system as follows:

$$\begin{aligned} \dot{x}_1 &= x_2 + (y^3 - y)/(1 + y^4) + 0.1\sin(0.1t) \\ \dot{x}_2 &= y^2 + \sin(y) + 0.1\cos(0.1t) + u \\ y &= x_1 \end{aligned} \tag{44}$$

where x_1, x_2 are system states, y and u are the output and input, respectively. The objective is for y to track the desired trajectory y_r, which is generated by a second-order filter $y_r = [w_n^2/(s^2 + 2\zeta w_n s + w_n^2)]y_{\mathrm{ref}}$ with $w_n = 1.5$, $\zeta = 0.8$, and for y_{ref} defined to be a square wave of amplitude $Y_0 = 0.5$, period $T = 20$ s.

If the initial compact set is chosen as $\Omega_y^0 := \{y \in R \,\big|\, |y| \leq k_0\}$, where $k_0 = 0.5$, we can specify another compact set $\Omega_y := \{y \in R \,\big|\, |y| \leq k_{c_1}\}$, where $k_{c_1} = 1.05 > k_0 + Y_0 + |y_r(0)| = 1.0$. Thus, we have that $k_{b_1} = k_{c_1} - A_0 = 0.55$.

The simulation results are shown in Figs. 3–5. Fig. 3 shows the output tracking performance. It can be seen that the output y remains within the compact set $\Omega_y := \{y \in \mathbb{R} \,\big|\, |y| \leq k_{c_1}\}$ and tracks the desired trajectory y_r to a neighborhood of zero when the proposed BLF-based control is used. The tracking error $z_1 = y - y_r$ and the control u are shown in Fig. 4. It is noted that there are some spikes in the control signal $u(t)$ at $t = nT/2 (n = 1, 2, \ldots)$. This is caused by the nonlinear term $\frac{z_1}{k_{b_1}^2 - z_1^2}$ in (24) and (25). For the square wave reference signal y_{ref}, there are some jumps at $t = nT/2 (n = 1, 2, \ldots)$, which result in peaks for the tracking error signal z_1. Before $z_1(t)$ approaches the barriers at $z_1 = \pm 0.55$, the nonlinear term $\frac{z_1}{k_{b_1}^2 - z_1^2}$ grows rapidly and leads to a large control effort that prevents z_1 from the barriers. It can be seen that z_1 remains in the set $|z_1| < k_{b_1}$ in Fig. 4, and thus, $|y| < k_{c_1}$, such that NN approximation is valid. In addition, Fig. 5 shows output trajectories for different initial conditions. It indicates that with the proposed BLF-based control, the output y, starting from a initial compact set $\Omega_y^0 := \{y \in R \,\big|\, |y| \leq k_0\}$, can always stay within the specified compact set $\Omega_y := \{y \in R \,\big|\, |y| \leq k_{c_1}\}$ for all time, which ensures that NN approximation is valid.

VI. Conclusion

In this brief, adaptive observer backstepping using NN has been presented for uncertain output feedback systems. The BLF has been incorporated into Lyapunov synthesis to address two open and challenging problems in the neuro-control area. The present approach would provide both theoretical criteria and practical insights for the design and implementation of NN-based control. It could be considered as a supplement or an improvement to the state of art in neuro-control field.

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Marginalized Neural Network Mixtures for Large-Scale Regression

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***Abstract*—For regression tasks, traditional neural networks (NNs) have been superseded by Gaussian processes, which provide probabilistic predictions (input-dependent error bars), improved accuracy, and virtually no overfitting. Due to their high computational cost, in scenarios with massive data sets, one has to resort to sparse Gaussian processes, which strive to achieve similar performance with much smaller computational effort. In this context, we introduce a mixture of NNs with marginalized output weights that can both provide probabilistic predictions and improve on the performance of sparse Gaussian processes, at the same computational cost. The effectiveness of this approach is shown experimentally on some representative large data sets.**

***Index Terms*—Bayesian models, Gaussian processes, large data sets, multilayer perceptrons, regression.**

I. Introduction

In recent years there has been a growing interest in Gaussian processes (GPs) as an alternative to neural networks (NNs) for regression tasks. GPs offer important advantages.

1) They provide full posterior probability density estimations. The posterior variance can be used to assess each prediction's uncertainty. Unlike traditional NNs, this predictive variance is not constant and depends on the test case being considered.
2) They use the "kernel trick," which reduces the modeling task to the selection of a moderate number of hyperparameters, which in turn reduces the risk of overfitting.
3) For some kernels, GPs correspond to predictors with NN structures in which all the weights have been integrated out (see Section II-A); therefore, all possible weight values are taken into account, in contrast to a single, best performing set of parameters, which is the standard, maximum likelihood form of work with NNs.

But all these advantages come at a cost: as we will see in Section II, GP model selection takes $\mathcal{O}(n^3)$ time (where n is the number of training samples). Probabilistic predictions can be made in $\mathcal{O}(n^2)$ time each, after some precomputations which take $\mathcal{O}(n^3)$ time. This cubic and quadratic scaling renders GPs impractical for real world applications with massive data sets, which are increasingly frequent.

To alleviate this problem, several approximations to GPs have been proposed. The current state-of-the-art approximation, denominated "sparse Gaussian processes using pseudo-inputs," was introduced in [10]. Later, it was renamed to fully independent training conditional (FITC) model in the unifying framework of [7]. This approximation achieves near full-GP performance at a much reduced cost: computing time is linear in n for learning and constant (independent of n) for making probabilistic predictions. We will review this approach in Section II-B.

In this brief, instead of simplifying a full GP, we try to achieve the advantages of GPs the other way around, applying Bayesian techniques to finite NNs: we marginalize out some of the parameters of an NN (the output weights) and then combine the predictions of several NNs. The resulting algorithm, marginalized NN mixture (MNNmix) has roughly the same computational cost as FITC. The idea of using priors on NN weights has been around for long, but most approaches do not marginalize them out analytically (see [6]).

The rest of the paper is organized as follows: In Section II, we will be reviewing the relevant portion of GPs for regression and their relation to infinite NNs and FITC. Marginalized NNs are introduced in Section III, and Section IV explains how to combine them. In Section V, we will test their performance on some large data sets, comparing MNNmix results with those provided by FITC and a full GP. Finally, we will provide a concluding discussion in Section VI.

II. Review of Gaussian Process Regression

Assume that we are given a set of independent and identically distributed (i.i.d.) samples $\mathcal{D} \equiv \{\mathbf{x}_j, y_j | j = 1, \dots, n\}$, where each D-dimensional input $\mathbf{x}_j$ is associated with a scalar output y_j. The regression task goal is, given a new input $\mathbf{x}_*$, to predict the corresponding output y_* based on $\mathcal{D}$.

The GP regression model assumes that the outputs can be modeled as some noiseless latent function of the inputs plus an independent noise

$$y = f(\mathbf{x}) + \varepsilon,$$

and then sets a zero-mean[1] GP prior on $f(\mathbf{x})$ and a Gaussian

[1] It is customary to subtract the sample mean from data $\{y_j\}_{j=1}^n$, which allows assumption of a zero-mean model.

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