Computing integral asymptotics using toric blow-ups of ideals

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Laplace Integrals

Laplace integrals of the form

\[ Z(N) = \int_{W} e^{-Nf(\omega)} \varphi(\omega) \, d\omega \]

occur frequently in machine learning, computational biology and combinatorics. Here, \( N \) is a positive real number, \( W \subset \mathbb{R}^d \) is a small nbhd of the origin, and \( f \) and \( \varphi \) are real-valued analytic functions where \( f \) attains its minimum at the origin.

We are often interested in approximating \( Z(N) \) for large \( N \).

Example:
In statistics, \( Z(N) \) may be a marginal likelihood integral used in model selection. We usually approximate such integrals using MCMC methods. In combinatorics, \( Z(N) \) may be the coefficient of a rational generating function, representing the count of an interesting combinatorial object.
Laplace Approximation

**Example**: Let $H(\omega)$ be the Hessian of $f$. If $H(0) \succ 0$ and $\varphi(0) > 0$, then asymptotically

$$Z(N) \approx e^{-Nf(0)} \cdot \varphi(0) \sqrt{\frac{(2\pi)^d}{\text{det } H(0)}} \cdot N^{-d/2} \quad \text{as } N \to \infty.$$  

Note that the integral asymptotics depend on the *geometry* of the function $f$ near its minimum points.

**Remark**: This formula is used to prove Stirling’s approximation.
Asymptotic Theory

More generally, even if $\det H(0) = 0$, Arnol’d–Guseĭn-Zade–Varchenko showed that asymptotically,

$$Z(N) \approx e^{-Nf(0)} \cdot CN^{-\lambda}(\log N)^{\theta-1}, \quad N \to \infty$$

for some positive $C \in \mathbb{R}$, $\lambda \in \mathbb{Q}$, $\theta \in \mathbb{Z}$. Here, $\lambda$ is the real log canonical threshold of $f$, and $\theta$ its multiplicity. We denote $\text{RLCT}(f; \varphi) := (\lambda, \theta)$.

**Theorem (AGV):**

The RLCT $\lambda$ of $f$ is the smallest pole of the zeta function

$$\zeta(z) = \int_{W} f(\omega)^{-z} \varphi(\omega)d\omega, \quad z \in \mathbb{C},$$

and $\theta$ is the multiplicity of this pole.

The poles of $\zeta(z)$ are computed using a resolution of singularities of $f$. 
Regularly Parametrized Functions

We were inspired by our statistical examples to study regularly parametrized analytic functions $f$, i.e. $f$ is a composition of maps

$$W \xrightarrow{g} U \xrightarrow{h} \mathbb{R}$$

where $W \subset \mathbb{R}^d$, $U \subset \mathbb{R}^k$ are small nbhds of the origin 0, $h$ attains its minimum uniquely at 0 and the Hessian of $h$ is positive definite at 0.

We also assume that $g$ is a polynomial map, and we want to exploit this polynomiality in our computations.

**Goal of this talk:**

- Show how to use *ideal-theoretic methods* to find a resolution of singularities for such functions $f$ and to compute its RLCT.
- Compute the *leading coefficient* $C$ in the asymptotics of $Z(N)$. 
Ideal-theoretic Methods
Sos-nondegeneracy

Let $[\omega^\alpha]f$ denote the coefficient of a monomial $\omega^\alpha$ in a polynomial $f$. Recall that $f$ is singular at $x \in \mathbb{R}^d$ if $f(x) = 0$ and $\nabla f(x) = 0$.

**Definitions (Varchenko):** Let $f \in \mathbb{R}[\omega]$ be a polynomial.

*Newton polyhedron* $\mathcal{P}(f) = \text{conv}\{\alpha \in \mathbb{R}^d : [\omega^\alpha]f \neq 0\}$.

Given $\gamma \subset \mathbb{R}^d$, *face polynomial* $f_\gamma = \sum_{\alpha \in \gamma} ([\omega^\alpha]f)\omega^\alpha$.

We say $f$ is **nondegenerate** if $f_\gamma$ is nonsingular at all $x \in (\mathbb{R}^\ast)^d$ for all compact faces $\gamma \in \mathcal{P}(f)$.

**Definitions (L.):** Let $I \subset \mathbb{R}[\omega]$ be an ideal.

*Newton polyhedron* $\mathcal{P}(I) = \text{conv}\{\alpha \in \mathbb{R}^d : [\omega^\alpha]f \neq 0 \text{ for some } f \in I\}$.

Given $\gamma \subset \mathbb{R}^d$, *face ideal* $I_\gamma = \langle f_\gamma : f \in I \rangle$.

We say $I$ is **sos-nondegenerate** if $f_1^2 + \ldots + f_r^2$ is nondegenerate for some generating set $\{f_1, \ldots, f_r\}$.

**Remark:** sos = sum-of-squares.
Sos-nondegeneracy

**Proposition (L.):**
If \( I = \langle f_1, \ldots, f_r \rangle \) and \( \gamma \subset \mathcal{P}(I) \) is a compact face, then \( I_\gamma = \langle f_1\gamma, \ldots, f_r\gamma \rangle \).

We have the following *equivalent definitions* of sos-nondegeneracy.

**Proposition (L.):**
1. For some generating set \( \{f_1, \ldots, f_r\} \), \( f_1^2 + \ldots + f_r^2 \) is nondegenerate.
2. For all generating sets \( \{f_1, \ldots, f_r\} \), \( f_1^2 + \ldots + f_r^2 \) is nondegenerate.
3. For all compact faces \( \gamma \subset \mathcal{P}(I) \), the real variety \( \mathcal{V}(I_\gamma) \) does not intersect the torus \((\mathbb{R}^*)^d\).

**Remark:** We discovered later that Saia has a notion of nondegeneracy similar to (3) for ideals in the ring of *complex* formal power series.

**Proposition (Zwiernik):**
Monomial ideals are sos-nondegenerate.
Let $f : W \xrightarrow{g} U \xrightarrow{h} \mathbb{R}, W \subset \mathbb{R}^d, U \subset \mathbb{R}^k$ be regularly parametrized. Suppose $g(0) = 0$, $h(0) = 0$, and the minimum of $h$ is attained at 0. Define the fiber ideal $\langle g_1(\omega), \ldots, g_k(\omega) \rangle$. It is the ideal of the fiber $g^{-1}(0)$.

Define $\text{RLCT}(I; \varphi)$ of an ideal $I = \langle g_1, \ldots, g_k \rangle$ to be the smallest pole and its multiplicity of the zeta function

$$\zeta(z) = \int_W (g_1^2 + \cdots + g_k^2)^{-z/2} \varphi(\omega)d\omega.$$ 

**Proposition (L):**

$\text{RLCT}(f; \varphi) = (\lambda/2, \theta)$, where $(\lambda, \theta) = \text{RLCT}(I; \varphi)$.

**Proposition (L.):**

Let $\rho : \mathcal{M} \to W$ be a principalization map for the fiber ideal $I$, i.e. the pullback $\rho^{-1}(I)$ is locally principal on $\mathcal{M}$. Then, $\rho$ desingularizes $f$. 
Toric Blowups

Let $\mathcal{F}$ be a smooth polyhedral fan supported on the positive orthant $\mathbb{R}^d_{\geq 0}$. [smooth: each cone is generated by a subset of some basis of $\mathbb{Z}^d$]

Recall that we can associate to $\mathcal{F}$, a toric variety $\mathbb{P}(\mathcal{F})$ covered by open affines $U_{\sigma} \simeq \mathbb{R}^d$, one for each maximal cone $\sigma$ of $\mathcal{F}$.

We also have a blowup map $\rho_\mathcal{F} : \mathbb{P}(\mathcal{F}) \rightarrow \mathbb{R}^d$ described by monomial maps $\rho_{\mathcal{F}, \sigma} : U_{\sigma} \rightarrow \mathbb{R}^d$, $\mu \mapsto \mu^\nu$, on the open affines. [The columns of the matrix $\nu$ are minimal generators of the maximal cone $\sigma$, and $(\mu^\nu)_i = \mu^{\nu_i}$ where $\nu_i$ is the $i$th row of $\nu$.]

**Proposition (L.):**
Given a fiber ideal $I$, let $\mathcal{F}$ be a smooth refinement of the normal fan of the Newton polyhedron $\mathcal{P}(I)$. If $I$ is sos-nondegenerate, then the toric blowup $\rho_\mathcal{F} : \mathbb{P}(\mathcal{F}) \rightarrow \mathbb{R}^d$ desingularizes $f$. 
Asymptotic Lower Bound

Given $\tau \in \mathbb{Z}^d_{\geq 0}$, define the $\tau$-distance $l_\tau$ of a polyhedron $\mathcal{P} \subset \mathbb{R}^d$ to be the smallest $t \geq 0$ such that $t(\tau_1 + 1, \ldots, \tau_d + 1) \in \mathcal{P}$, and its multiplicity $\theta_\tau$ to be the codim of the face of $\mathcal{P}$ at this intersection.

Let $I = \langle x^4, x^2y, xy^3, y^4 \rangle$. 
Asymptotic Lower Bound

Given $\tau \in \mathbb{Z}_{\geq 0}^d$, define the $\tau$-distance $l_\tau$ of a polyhedron $\mathcal{P} \subset \mathbb{R}^d$ to be the smallest $t \geq 0$ such that $t(\tau_1 + 1, \ldots, \tau_d + 1) \in \mathcal{P}$, and its multiplicity $\theta_\tau$ to be the codim of the face of $\mathcal{P}$ at this intersection.

Let $I = \langle x^4, x^2y, xy^3, y^4 \rangle$.

For $\tau = (0, 0)$: $l_\tau = 8/5$, $\theta_\tau = 1$. 
Asymptotic Lower Bound

Given \( \tau \in \mathbb{Z}_{\geq 0}^d \), define the \( \tau \)-distance \( l_\tau \) of a polyhedron \( \mathcal{P} \subset \mathbb{R}^d \) to be the smallest \( t \geq 0 \) such that \( t(\tau_1 + 1, \ldots, \tau_d + 1) \in \mathcal{P} \), and its multiplicity \( \theta_\tau \) to be the codim of the face of \( \mathcal{P} \) at this intersection.

Let \( I = \langle x^4, x^2y, xy^3, y^4 \rangle \).

For \( \tau = (0, 0) \) : \( l_\tau = 8/5, \theta_\tau = 1 \).

For \( \tau = (1, 0) \) : \( l_\tau = 1, \theta_\tau = 2 \).
Asymptotic Lower Bound

Given \( \tau \in \mathbb{Z}_{\geq 0}^d \), define the \( \tau \)-distance \( l_\tau \) of a polyhedron \( \mathcal{P} \subset \mathbb{R}^d \) to be the smallest \( t \geq 0 \) such that \( t(\tau_1 + 1, \ldots, \tau_d + 1) \in \mathcal{P} \), and its multiplicity \( \theta_\tau \) to be the codim of the face of \( \mathcal{P} \) at this intersection.

**Theorem (L.):**
Given a regularly parametrized function \( f \) and a vector \( \tau \in \mathbb{Z}_{\geq 0}^d \), let \( I \) be the fiber ideal and let \( (l_\tau, \theta_\tau) \) be the \( t \)-distance and its multiplicity of the Newton polyhedron \( \mathcal{P}(I) \). Then, asymptotically

\[
Z(N) = \int_W e^{-Nf(\omega)}\omega^\tau d\omega
\]

is **bounded below** by \( CN^{-1/(2l_\tau)}(\log N)^{\theta_\tau - 1} \) for some constant \( C \). This bound is tight if the fiber ideal \( I \) is sos-nondegenerate.

In other words, if \( I \) is sos-ndg, then \( RLCT(I; \omega^\tau) = (1/l_\tau, \theta_\tau) \).
[This is the **real analog** of Howald’s result for complex LCTs.]
Leading Coefficients
We want to compute the leading coefficient $C$ in the asymptotics

$$Z(N) = \int_{[0, \varepsilon]^d} e^{-Nf(\omega)} \omega^\tau d\omega \approx CN^{-\lambda} (\log N)^{\theta-1}.$$ 

where $f$ is nondegenerate, $\varepsilon$ is sufficiently small and $\tau \in \mathbb{Z}_{\geq 0}^d$.

Because $f$ is nondegenerate, any smooth refinement of the normal fan of the Newton polyhedron $\mathcal{P}(f)$ desingularizes $f$ at the origin. We fix $\mathcal{F}$ to be one such refinement.

We pick $\varepsilon$ to be sufficiently small so that under the blowup $\rho_\mathcal{F}$, the strict transform $g$ of $f$ is positive at every point in $\rho_\mathcal{F}^{-1}[0, \varepsilon]^d$.

By scaling the coordinates $\omega$, we may assume for simplicity that $\varepsilon = 1$. 

**Preliminaries**
Recall that \((\lambda, \theta) = (1/l_\tau, \theta_\tau)\) where \(l_\tau\) is the \(\tau\)-distance of the Newton polyhedron \(\mathcal{P}(f)\) and \(\theta_\tau\) its multiplicity.

Let \(\sigma_\tau\) be the cone in the normal fan of \(\mathcal{P}(f)\) corresponding to the face at this intersection. Note that \(\sigma_\tau\) has dimension \(\theta\).

In the refinement \(\mathcal{F}\), we consider the set \(\mathcal{F}_\tau\) of all maximal cones which intersect \(\sigma_\tau\) in dimension \(\theta\). For each \(\sigma\) in \(\mathcal{F}_\tau\), let \(\nu\) be the matrix whose columns are the minimal generators of \(\sigma\) and where the first \(\theta\) columns are generators of \(\sigma_\tau\).
Leading Coefficient

**Theorem (L.):**
The leading coefficient $C$ in the asymptotics of $Z(N)$ equals

$$\frac{\Gamma(\lambda)}{(\theta - 1)!} \sum_{\sigma \in \mathcal{F}_\tau} \prod_{i=1}^{\theta} (\nu \alpha)_i^{-1} \int_{[0,1]^{d-\theta}} g(0, \bar{\mu})^{-\lambda \bar{\mu} m^{-1}} d\bar{\mu}.$$ 

Here, $\Gamma(\cdot)$ is the Gamma function, and for each $\sigma$ in $\mathcal{F}_\tau$,
- $\nu$ is the matrix of minimal generators,
- $\alpha \in \mathcal{P}(f)$ is the vertex dual to $\sigma$,
- $\mu = (\hat{\mu}, \bar{\mu}) \in \mathbb{R}^\theta \times \mathbb{R}^{d-\theta}$,
- $m = \nu(-\lambda \alpha + \tau + 1) = (\hat{m}, \bar{m}) \in \mathbb{R}^\theta \times \mathbb{R}^{d-\theta}$, and
- $g(\hat{\mu}, \bar{\mu}) = f(\mu^\nu) \mu^{-\nu \alpha}$ is the strict transform of $f$ in the open affine $U_\sigma$.

**Work in Progress:** Macaulay2 code which implements this formula.
Example

**Question:** Find the first term asymptotics of the integral

\[ Z(N) = \int_{[0,1]^2} (1 - x^2 y^2)^{N/2} \, dx dy. \]

[This question comes from a statistical example involving coin tosses.]

**Solution:** Rewrite the integral as \( Z(N) = \int_{[0,1]^2} e^{-Nf(x,y)} \, dx dy \) where

\[ f(x, y) = -\frac{1}{2} \log(1 - x^2 y^2). \]

Here, \( f \) is regularly parametrized, because it is the composition of maps \((x, y) \mapsto xy \) and \( t \mapsto -\frac{1}{2} \log(1 - t^2). \) The fiber ideal \( I = \langle xy \rangle \) is monomial and sos-nondegenerate. Thus, \( f \) is nondegenerate and the Newton polyhedron \( \mathcal{P}(f) \) is the orthant cornered at \((2, 2)\). Using our formula, we get

\[ Z(N) \approx \sqrt{\frac{\pi}{8}} \, N^{-1/2} \log N. \]
In fact, using similar techniques, we get the asymptotic expansion
\[ Z(N) \approx \sum_{\lambda, \theta} C_{\lambda, \theta} N^{-\lambda} (\log N)^{\theta - 1} \]
where the first few terms are
\[
C_{\frac{1}{2}, 2} = \sqrt{\frac{\pi}{8}}, \quad C_{\frac{1}{2}, 1} = -\sqrt{\frac{\pi}{8}} \left( \frac{1}{\log 2} - 2 \log 2 - \gamma \right),
\]
\[
C_{1, 2} = -\frac{1}{4}, \quad C_{1, 1} = \frac{1}{4} \left( \frac{1}{\log 2} + 1 - \gamma \right),
\]
\[
C_{\frac{3}{2}, 2} = -\frac{\sqrt{2\pi}}{128}, \quad C_{\frac{3}{2}, 1} = \frac{\sqrt{2\pi}}{128} \left( \frac{1}{\log 2} - 2 \log 2 - \frac{10}{3} - \gamma \right),
\]
\[
C_{2, 2} = 0, \quad C_{2, 1} = -\frac{1}{24}.
\]

Here, \( \gamma \) is the Euler-Machheroni constant
\[
\gamma = \lim_{n \to \infty} \left( \sum_{i=1}^{n} \frac{1}{k} - \log n \right) \approx 0.5772156649.
\]
“Algebraic Methods for Evaluating Integrals in Bayesian Statistics”

http://math.berkeley.edu/~shaowei/swthesis.pdf

(PhD dissertation, May 2011)
References